Dynamics of Homogenous Magnetizations in Strong Transverse Driving Fields

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January 21, 1998

Abstract

Spatially homogeneous solutions of the Landau–Lifshitz–Gilbert equation are analysed. The different cases of conservative as well as dissipative motion are considered separately. For the linearly polarized driven Hamiltonian system we apply a global perturbation theory to uncover the main resonances as well as the phase space structure. The case of circularly polarized driven dissipative motion is studied in detail. We present the complete bifurcation diagram including bifurcations up to codimension three.

PACS: 03.20, 02.30

1 Introduction

The investigation of strongly driven ferromagnetic systems is an interesting field of research especially from the point of view of nonlinear dynamics. The fundamental equation of motion, which governs the dynamics of a macroscopic magnetization has been proposed already by Landau and Lifshitz [1]

$$\dot{\mathbf{S}} = -\mathbf{S} \times (\mathbf{H}_{eff}(\mathbf{S},t) + \Gamma \mathbf{S} \times \mathbf{H}_{eff}(\mathbf{S},t)) \quad . \tag{1}$$

It describes the motion of a macroscopic magnetization $\mathbf{S}$ under the influence of an effective local magnetic field $\mathbf{H}_{eff}(\mathbf{S},t)$. The second term in this equation, frequently called the Gilbert damping, represents the dominant contribution among several dissipative terms, which arise in microscopic derivations of this equation (e.g. ref.[2, 3]). It is a special feature of eq.(1) that it preserves the modulus of the magnetization. In order to give eq.(1) a definite meaning one has to specify the effective field. Usually contributions from external fields, dipolar and exchange interactions, anisotropy etc. are taken into account. They turn eq.(1) in general into a complicated partial differential equation which may even be nonlocal in space. Hence it is impossible to discuss the dynamical behaviour in general. We will focus in our article on the analysis of spatially homogeneous states. Even this seemingly simple problem turns out to be highly nontrivial.

To begin with the effective field has to be specified. We take a static external field, a time dependent transversal driving field and an uniaxial anisotropy parallel to the static field into account while contributions from the exchange interaction obviously vanish and the dipolar interaction may be thought to be contained in the anisotropy term [4]. Under these conditions eq.(1) becomes a two dimensional system of nonautonomous ordinary differential equations, which is Hamiltonian or dissipative depending on the value of the damping constant. We will treat both cases in our paper independently but mainly focus on the analysis of the dissipative motion.

Low dimensional conservative spin systems provide physical textbook examples of Hamiltonian dynamics. Discussions of such systems can be found in the literature. For a treatment of undriven systems consult [5, 6]. Furthermore Frahm and Mikeska have shown [7] that the equations of motion are integrable, if a circularly polarized driving field is applied to a single spin. For linearly polarized driving fields their numerical simulations indicate that the system is not integrable. In contrast to their approach we will describe the Hamiltonian dynamics analytically using a perturbation scheme which yields the global phase space structure.

On the other hand a systematic analysis of driven dissipative spin systems does not seem to be available in the literature. It is our main concern to describe the damped spin system under the influence of a rotating driving field. In that case the system becomes autonomous using a transformation into a rotating coordinate system. Although such a system cannot become chaotic we will show by performing a careful bifurcation analysis that highly nontrivial phenomena occur.

2 Hamiltonian Dynamics under Linearly Polarized Pumping

Undamped spin systems provide a nontrivial example of simple Hamiltonian systems [7]. It is well known that an axisymmetric system is integrable if a rotating driving field is applied. Hence we focus in this section on systems which are driven by linearly polarized fields. They give rise to very complicated dynamics. To be definite the effective field is chosen as

$$\mathbf{H}_{eff}(\mathbf{S},t) = (H_z - aS_z)\,\mathbf{e}_z + h_\perp \cos(\omega t)\,\mathbf{e}_x \quad . \tag{2}$$

Spin systems of this type have been discussed already in the literature. But in contrast to former approaches [7, 8] a static field is incorporated in our treatment which breaks the plane symmetry $S_z \to -S_z$. For that reason the methods employed in these references cannot be applied to our equations of motion.

The dynamical system (1) subjected to the effective field (2) is described by the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \varepsilon\,\mathcal{H}_1 \\ &= S_z\left(H_z - \frac{aS_z}{2}\right) + \varepsilon\, h_\perp \sqrt{1-S_z^2}\ \cos(\varphi)\ \sin(\omega t) \quad . \end{aligned} \tag{3}$$

Here canonical coordinates $S_x + iS_y = \sqrt{1-S_z^2}\exp(i\varphi)$ and the formal expansion parameter ε have been introduced for convenience. The unperturbed i.e. the undriven system, $\varepsilon = 0$, is obviously integrable. For later reference its frequency is denoted by

$$\Omega(S_z) := H_z - aS_z \quad . \tag{4}$$

We will apply a global perturbation theory for weak driving fields to uncover the phase space structure of our system. The calculations will be confirmed by numerical simulations.

2.1 Global Perturbation Theory

In order to get an overview of the phase portrait of the perturbed Hamiltonian system we follow a method which has been proposed by Dunnet at al.[9]. Its main idea is quite simple and is based on the construction of an approximate constant of motion. For the sake of completeness we first recall some technical details which are formulated in terms of a Lie technique. It allows to perform higher order perturbation expansions in a systematic way (cf. ref.[10] section 2.5).

One is looking for a canonical transformation $x = (S_z, \varphi) \rightarrow \bar{x} = (\bar{S}_z, \bar{\varphi})$ which transforms the Hamiltonian (3) into an integrable one at least up to a certain order in the expansion parameter ε. If $w(J, \varphi, \varepsilon)$ denotes the generator of the infinitesimal transformation it obeys

$$\frac{d\bar{x}}{d\varepsilon} = [\bar{x}, w] \tag{5}$$

where $[.,.]$ is the Poisson bracket. The evolution operator $\bar{x}(\varepsilon) = \mathbf{T}x$ of this formal Hamiltonian system with "*time*" ε defines the desired transformation. With the help of the Lie operator $\mathbf{L} = [w, \quad]$ eq.(5) can be cast into the form

$$\frac{d\mathbf{T}}{d\varepsilon} = -\mathbf{T}\mathbf{L}, \qquad \mathbf{T}(\varepsilon = 0) = \mathbf{1} \quad . \tag{6}$$

This algebraic version of the transformation is well suited for perturbation expansions. Using the formal power series

$$\mathbf{L} = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{L}_{n+1}, \qquad \mathbf{T} = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{T}_n, \qquad \mathbf{T}^{-1} = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{T}_n^{-1} \tag{7}$$

eq.(6) can be solved for $\mathbf{T}^{-1}$

$$\mathbf{T}_n^{-1} = -\frac{1}{n} \sum_{m=0}^{n-1} \mathbf{L}_{n-m} \mathbf{T}_m^{-1} \quad . \tag{8}$$

The generator w is now specified by the requirements that it does not contain secular contributions and that the Hamiltonian of the transformed system $\bar{\mathcal{H}}$ depends solely on the new action variable keeping the new system integrable. These conditions together with the formal series expansions

$$\mathcal{H} = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{H}_n, \qquad \bar{\mathcal{H}} = \sum_{n=0}^{\infty} \varepsilon^n \bar{\mathcal{H}}_n \tag{9}$$

lead to the recurrence relations (cf. ref.[10] for further details)

$$\frac{\partial w_n}{\partial t} + [w_n, \mathcal{H}_0] = n \left(\bar{\mathcal{H}}_n - \mathcal{H}_n \right) - \sum_{m=1}^{n-1} \left(\mathbf{L}_{n-m} \bar{\mathcal{H}}_m + m \mathbf{T}_{n-m}^{-1} \mathcal{H}_m \right) \quad . \tag{10}$$

Owing to the fact, that the unperturbed motion can be integrated explicitly these equations can be solved successively. The formal transformation generated in this manner contains small denominators however which carry over to the new Hamiltonian $\bar{\mathcal{H}}$ also. Hence neither $\bar{\mathcal{H}}$ itself nor the bare action $\bar{S}_z$ may serve as the global invariant we are looking for. But as an arbitrary function $\bar{f}$ of the new action $\bar{S}_z$ is a constant of motion also it turns out to be possible to avoid the singularities just mentioned in every order of the perturbation expansion by a suitable choice for $\bar{f}$.

Going back to the original variables it is finally the counter image $\mathbf{T}^{-1}\bar{f}$, being a constant of motion also, which contains the topology of the phase space.

To be definite we perform the expansion up to second order. For $n = 1$ eq.(10) reads explicitly

$$\frac{\partial w_1}{\partial t} + [w_1, \mathcal{H}_0] = \bar{\mathcal{H}}_1 - h_\perp \sqrt{1 - S_z^2}\ \cos(\varphi)\ \sin(\omega t) \qquad . \tag{11}$$

With the choice $\bar{\mathcal{H}}_1 = 0$ we find for the generator

$$w_1 = h_\perp \sqrt{1 - S_z^2} \left(\frac{\cos(\varphi + \omega t)}{2(\Omega + \omega)} - \frac{\cos(\varphi - \omega t)}{2(\Omega - \omega)} \right) \quad . \tag{12}$$

It has only two small denominators which reflect the resonances of first order at $\Omega(S_z) = \pm\omega$. Higher harmonics do not occur in the present case because of the special nature of the coupling of the driving field. The second order equation is given by

$$\begin{aligned} \frac{\partial w_2}{\partial t} + [w_2, \mathcal{H}_0] &= 2\,\bar{\mathcal{H}}_2 - h_\perp^2 a \frac{(\cos(2\varphi) + \cos(2\omega t) - 1)(1 - S_z^2)(\Omega^2 + \omega^2)}{(2(\Omega^2 - \omega^2))^2} \\ &\quad + h_\perp^2 \frac{S_z(1 - \cos(2\omega t))\,\Omega}{(2(\Omega^2 - \omega^2))} \\ &\quad - h_\perp^2 a \left(1 - S_z^2\right) \left(\frac{\cos(2\varphi + 2\omega t)}{8(\Omega + \omega)^2} + \frac{\cos(2\varphi - 2\omega t)}{8(\Omega - \omega)^2} \right) \qquad . \end{aligned} \tag{13}$$

The choice

$$\bar{\mathcal{H}}_2 = -h_\perp^2 a \frac{(1 - S_z^2)(\Omega^2 + \omega^2)}{8(\Omega^2 - \omega^2)^2} + h_\perp^2 \frac{S_z \Omega}{4(\Omega^2 - \omega^2)} \tag{14}$$

ensures that the solution possesses no secular terms. The generator becomes

$$\begin{aligned} w_2 &= -h_\perp^2 \frac{\Omega\, S_z \sin(2\omega t)}{4\omega(\Omega^2 - \omega^2)} + h_\perp^2 a \frac{(1 - S_z^2)(\Omega^2 + \omega^2)\sin(2\varphi)}{8\Omega(\Omega^2 - \omega^2)^2} \\ &\quad + h_\perp^2 a \frac{(1 - S_z^2)(\Omega^2 + \omega^2)\sin(2\omega t)}{8\omega(\Omega^2 - \omega^2)^2} \\ &\quad + h_\perp^2 a \left(1 - S_z^2\right) \left(\frac{\sin(2\varphi - 2\omega t)}{16(\Omega - \omega)^3} + \frac{\sin(2\varphi + 2\omega t)}{16(\Omega + \omega)^3} \right) \quad . \end{aligned} \tag{15}$$

Clearly an additional resonant small denominator appears at $\Omega(S_z) = 0$. Now eqs.(7) tell us that the desired constant of motion results from

$$f = \mathbf{T}^{-1}\bar{f} = \bar{f} + \varepsilon\left[w_1, \bar{f}\right] + \frac{\varepsilon^2}{2}\left(\left[w_2, \bar{f}\right] + \left[w_1, \left[w_1, \bar{f}\right]\right]\right) \quad . \tag{16}$$

In general eq.(16) contains singularities at the resonances $\Omega(S_z) = 0, \pm\omega$. If however the function $\bar{f}$ is chosen in such a way that the zeroes of its derivative cancel all these poles, f remains regular in the whole phase space. The authors of ref.[9] suggested this expression to be a reasonable approximation for a global invariant of the system. In principle one can use any function $\bar{f}$ which satisfies the above mentioned constraint. For graphical purposes it is however convenient to choose a function whose numerical values are of the same order of magnitude in a large region of the phase space. We therefore start from the following expression

$$\bar{f}(\bar{S}_z) = \int_0^{\bar{S}_z} \cos^3\left(\frac{\pi\Omega(x)}{2\omega}\right)\Omega(x)\,dx \quad . \tag{17}$$

Combining eqs.(12), (15), and (16) the invariant up to second order perturbation theory is obtained after some tedious calculation. We refrain from writing down the lengthy result explicitly.

The level lines of the invariant constructed in this manner provide a good approximation of the phase space structure. We give plots of these level lines at fixed time t which is equivalent to a Poincare section at the same time. Fig.1 shows the lines for the parameter values $\varepsilon h_\perp = 0.01$, $a = 1$, $H_z = 0.5$ and $\omega = 0.2$. Resonances of first and second order are clearly visible. Additionally second order islands can be recognized near the first order resonance. But these structures are reproduced only qualitatively because they depend on the expansion parameter in a highly non-analytical way (cf. ref.[11]). If the amplitude of the driving field is increased the level lines which cross the phase space are destroyed. Fig.2 shows an example for the parameter values $h_\perp = 0.025$, $a = 1$, $H_z = 0.5$ and $\omega = 0.2$. This effect may be attributed to an overlap of low order resonances and signals the destruction of KAM tori. However higher order perturbation calculations would be necessary to obtain more reliable results. < Fig.1 < Fig.2

2.2 Numerical Results

We confirm the analytical perturbational calculations of the previous section by direct numerical simulations of the full Hamiltonian dynamics. For that purpose the system has been integrated by using the subroutine D02BAF of the NAG library. Poincare plots have been computed by integrating over 100 periods of the driving field starting from 400 initial conditions distributed uniformly at $\varphi = 0, \pm\pi/2, \pi$. Fig.3 shows the Poincare plot for the parameter values used in Fig.1. Good agreement is observed. Especially the first and second order resonances are reproduced quantitatively by the perturbation expansion. But e.g. third order resonances are found in the numerical simulation being not incorporated in the perturbative approach which is of second order only. In addition the secondary islands near the first order resonances are overestimated by the perturbational approach. Fig.4 contains data at the parameter values used in Fig.2. Obviously KAM tori are destroyed by resonance overlap. The agreement between the perturbation expansion (cf. Fig.2) and the numerical simulation is merely qualitative. It signals that the validity of the expansion breaks down if the system is chaotic in large regions of the phase space. < Fig.3 < Fig.4

3 The Circularly Polarized Driven Dissipative System

We will now study the dynamics of the dissipative system which is fundamentally different from the case analysed in the previous section. It is our intention to give a rather complete survey over the dynamical behaviour in a simple but highly nontrivial situation. We consider a system being driven by a rotating transversal field. The effective field reads

$$\mathbf{H}_{eff}(\mathbf{S},t) = (H_z - aS_z)\,\mathbf{e}_z + h_\perp\left(\cos(\omega t)\,\mathbf{e}_x + \sin(\omega t)\,\mathbf{e}_y\right) \qquad . \tag{18}$$

The explicit time dependence of the equations of motion (1) may be eliminated using a transformation to a rotating frame because of the chiral symmetry of the dynamics. We obtain the autonomous equations of motion

$$\frac{d\mathbf{S}}{dt} = -\mathbf{S}\times\left(\mathbf{H}_{eff} + \Gamma\,\mathbf{S}\times(\mathbf{H}_{eff} + \omega\mathbf{e}_z)\right) \tag{19}$$

where

$$\mathbf{H}_{eff} = (H_z - aS_z - \omega)\,\mathbf{e}_z + h_\perp\mathbf{e}_x \tag{20}$$

denotes the time independent effective field in the rotating frame. It is the physical nature of the driving field which is responsible for the different magnetic fields in the reversible and the Gilbert term. This structure prevents eq.(19) from possessing a Lyapunov function and ultimately causes the complex behaviour of its solutions. On the other hand the modulus of the magnetization is preserved in the course of time so that the phase space is the two dimensional sphere. This fact enables us to keep the discussion to a great extent analytical but at the same time prevents the system from becoming chaotic.

As a constant factor can be absorbed into the time scale we restrict the subsequent treatment to the case $|a| = 1$ without loss of generality. We will start our analysis with a brief discussion of the fixed points. Subsequently local and global bifurcations will be analysed which will result in a complete overview of the dynamics of the system.

3.1 The Fixed Points

Even the calculation of the fixed points gives rise to a system of algebraic equations in general which cannot be solved without resorting to numerical methods. But our fixed point problem (cf. eq.(19)) can be reduced to a single algebraic equation of fourth order

$$\left(1 - S_z^2\right)\left((aS_z - \Delta)^2 + \gamma^2 h_\perp^2 S_z^2\right) - h_\perp^2 S_z^2 = 0 \quad . \tag{21}$$

Here the new parameters

$$\Delta := H_z - \frac{\omega}{1+\Gamma^2} \quad (22)$$

$$\gamma := \frac{\omega}{h_\perp}\frac{\Gamma}{1+\Gamma^2} \quad (23)$$

have been introduced and will be used in the sequel. The remaining components are determined via

$$S_x = \frac{\Delta - aS_z}{S_z h_\perp}\left(1 - S_z^2\right) \quad (24)$$

$$S_y = -\gamma\left(1 - S_z^2\right) \quad . \quad (25)$$

It is worthwhile to mention that because of eq.(21) the system has at least two and at most four fixed points. The explicit discussion is postponed to the next sections.

3.2 Method of Analysis

Before we enter the discussion of the various bifurcation diagrams it is useful to give a brief survey over the symmetries of the system under consideration. It is easy to check that the equations of motion are invariant with respect to the following two transformations

$$\mathbf{T}_1: \quad \Delta \to -\Delta \quad \gamma \to -\gamma \quad a \to -a \quad h_\perp \to -h_\perp \quad t \to -t \quad (26)$$

$$\mathbf{T}_2: \quad \Delta \to -\Delta \quad \gamma \to -\gamma \quad S_z \to -S_z \quad S_y = -S_y \quad . \quad (27)$$

Taking these symmetries into account we can restrict the following bifurcation analysis to the case $a = -1$ and $\gamma > 0$. In addition we will not analyse the full four dimensional parameter space (Γ, Δ, γ, $h_\perp$). We restrict ourselves to the fixed value of the damping constant $\Gamma = 0.1$.

Let us first describe the procedure which is applied to treat the local codimension one bifurcations (Hopf and saddle node bifurcations). It is convenient to perform this analysis by referring explicitly to the fact that the phase space is two dimensional. We rewrite eq.(19) in terms of the complex variable $Z = (S_x + iS_y)/(1+S_z)$. This corresponds to the well known stereographic projection of the sphere to the plane (cf. ref.[12])

$$\frac{dZ}{dt} = f(Z) := (i - \Gamma)\left(i\gamma h_\perp Z + \Delta Z - aZ\frac{1-|Z|^2}{1+|Z|^2} - \frac{h_\perp}{2}\left(1-|Z|^2\right)\right) \quad . \quad (28)$$

Saddle node bifurcations of the fixed points are determined by a single vanishing eigenvalue of the linearized system i.e. by the equations[1]

$$f(Z) = 0, \qquad \det(Df(Z)) = 0 \quad . \quad (29)$$

[1]Inspection of the polynomial eq.(21) shows that this condition coincides with a degeneracy of its zeroes.

The solutions of this set of algebraic equations define codimension one manifolds in parameter space on which saddle node bifurcations take place. With the help of the program PITCON [13] these manifolds can be computed easily by a continuation technique. In the same way Hopf bifurcations are detected. Owing to the dimensionality of the phase space they are determined by the equations

$$f(Z) = 0, \qquad \mathrm{Tr}(Df(Z)) = 0 \quad . \tag{30}$$

In addition using a well known formula for the coefficient of the third order in the Hopf normal form [14] we decide whether the bifurcation is sub– or supercritical. We stress that our explicit knowledge of the number of fixed points enables us to detect all local codimension one bifurcations.

The discussion of the local codimension one bifurcations will show that our system has at most one saddle point. Homoclinic bifurcations occurring eventually in connection with this point have been investigated using a numerical computation of the distance between its stable and unstable manifolds. This method seems to be more suitable than the quest for limit cycles as suggested by Doedel (ref.[15]). The respective codimension one bifurcation sets are again computed via PITCON.

Finally we have concentrated on the investigation of saddle node bifurcations of limit cycles. It is well known that the proper bifurcation manifold is born in a degenerated Hopf bifurcation. Using the program AUTO [15] we have been able to compute the saddle node bifurcation manifold of limit cycles also.

3.3 Bifurcation Diagrams

The results of our investigations are summarized in Figs.5,8–12. In order to get insight into the structure of the full three dimensional parameter space we show cross sections for several values of $h_\perp$. Let us first describe in detail the situation for $h_\perp = 0.1$ (cf. Fig.5) . Apart from the codimension one manifolds we find the $<$ Fig.5
following bifurcations[2] of codimension two:

- two cusp points at C_1 and C_2
- one degenerated Hopf bifurcation at H where a saddle node bifurcation for limit cycles meets a Hopf bifurcation
- two Arnold–Takens–Bogdanov bifurcations at A_1 and A_2 where saddle node, Hopf, and homoclinic bifurcations meet
- two saddle node connections at S_1 and S_2

[2] For a detailed explanation of these basic bifurcations the reader is referred to [14]

- one degenerated homoclinic bifurcation[3] at B where a saddle node bifurcation line of limit cycles meets a homoclinic bifurcation. The lines to the left of B cannot be resolved on the scale of Fig. 5 (cf. the small insert).

The local and global codimension one lines divide the parameter space in ten regions. The typical dynamical behaviour in each region and on the boundaries is described in the sequel:

- In region I there exist only one stable and one unstable fixed point.

– The saddle node bifurcation line, which separates region I from region II, generates a saddle point and a stable fixed point.

- Region II contains four fixed points, one saddle point, one unstable fixed point and two stable fixed points.

– The saddle node bifurcation line, which separates region II from region III, destroys a stable fixed point and the saddle point.

- In region III there exist one stable and one unstable fixed point.

– The subcritical Hopf bifurcation line, which separates region III from region IV, generates an unstable limit cycle. The unstable fixed point becomes stable.

- region IV contains two stable fixed points and one unstable limit cycle.

– The subcritical Hopf bifurcation line, which separates region IV from region I destroys the unstable limit cycle. One stable fixed point becomes unstable.

– The supercritical Hopf bifurcation line, which separates region IV from region V, generates a stable limit cycle and turns one stable fixed point into an unstable fixed point.

- In region V there exist a stable and an unstable fixed point and a stable and an unstable limit cycle.

– The limit cycles just mentioned are destroyed at the saddle node bifurcation line for limit cycles, which separates region V from region I.

– The saddle node bifurcation line, which separates region V from region VI (cf. the small insert in Fig.5), generates a stable fixed point and a saddle point. Moreover this bifurcation destroys the stable limit cycle.

- Region VI contains four fixed points (one saddle, one unstable and two stable fixed points) and an unstable limit cycle.

[3]For a description of this bifurcation see e.g. [16].

- The homoclinic bifurcation line, which separates region VI from region II, destroys the unstable limit cycle.

- Region II contains the same fixed points as region VI but no limit cycle.

- The saddle node bifurcation line, which separates region VI from region IV, destroys an unstable fixed point and the saddle point.

- The subcritical Hopf bifurcation line, which separates region VI from region VII, destroys the unstable limit cycle and turns a stable fixed point unstable.

- Region VII contains only four fixed points (one saddle, two unstable and one stable fixed point).

- The saddle node bifurcation line, which separates region VII from region III, destroys an unstable fixed point and the saddle point.

- The saddle node bifurcation line, which separates region V from region VIII, generates a stable fixed point and a saddle point.

- Region VIII contains four fixed points (one saddle point, one unstable and two stable fixed points) and two limit cycles (one stable and one unstable)

- The homoclinic bifurcation line, which separates region VIII from region VI, destroys the stable limit cycle.

- The supercritical Hopf bifurcation line, which separates region VIII from region IX, destroys the stable limit cycle and turns an unstable fixed point stable.

- Region IX contains four fixed points (one saddle and three stable fixed points) and one unstable limit cycle.

- The saddle node bifurcation line, which separates region IX from region IV, destroys a stable fixed point and the saddle point.

- The homoclinic bifurcation line, which separates region X from region VI, generates a stable limit cycle.

- Region X contains the same fixed points as region VI but two limit cycles (one stable and one unstable).

- The saddle node bifurcation line for limit cycles, which separates region X from region II destroys both limit cycles contained in region X.

Phase portraits for typical parameter values in the regions II and V–IX are displayed in Fig.6. <Fig.6

Our bifurcation diagram has an intricate structure especially as far as the two homoclinic bifurcation lines $\overline{A_1S_1}$ and $\overline{A_2S_2}$ are concerned. The insert of Fig.5 which resolves these curves on a finer scale shows that they end up in different points S_1 and S_2. This is a consequence of the fact that the two bifurcations in question play essentially different rolls with respect to the global phase portrait of the system (cf. Fig.7). Fig.7a shows a phase portrait on the line $\overline{A_1S_1}$. One realizes that the < Fig.7
homoclinic orbit encloses one unstable fixed point. Fig.7b shows the same phase portrait on the other homoclinic bifurcation line $\overline{A_2S_2}$. In this case the homoclinic orbit encircles two fixed points. Both situations cannot be deformed smoothly into each other and are therefore topological distinct[4]. We add that in Figs.7a and 7b the stable manifold of the saddle point may be thought of as employing just its two different branches of the unstable manifold in building the two homoclinic orbits.

In addition our bifurcation diagram reflects the topology of the phase space. For example region IV is bounded by two Hopf bifurcation lines. The upper line generates an unstable limit cycle. It is destroyed if one crosses the lower subcritical Hopf bifurcation line. As both Hopf lines are connected to different Arnold–Takens–Bogdanov points the fixed points involved in the Hopf bifurcations are different. Hence the limit cycle wanders continuously from one fixed point to the other. Such a scenario is impossible if the phase space is a plane, e.g. for a mechanical oscillator.

If we change the third bifurcation parameter $h_\perp$ the bifurcation diagram changes smoothly. However some qualitative modifications occur. Using the same notations as in the previous analysis we describe the major differences. Increasing $h_\perp$ the Hopf bifurcation line which begins at A_2 flattens and finally divides region VIII and IX into two parts (cf. Fig.8). In contrast to regions VIIIa and IXa regions VIIIb <Fig.8
and IXb contain an additional unstable limit cycle. At slightly higher values of the driving field $h_\perp$ the same phenomenon occurs in region V (cf. Fig.9). <Fig.9

A dramatic change in the bifurcation diagram occurs at the Arnold–Takens–Bogdanov point A_2 by increasing the driving field further. Fig.10 shows the diagram <Fig.10
for $h_\perp = 0.8$. The major differences to the previous case (cf. Fig.9) are

- The Hopf bifurcation line starting at the point A_2 has become supercritical.
- The homoclinic bifurcation line lies above the Hopf bifurcation line.
- The saddle node bifurcation for limit cycles now ends up in a degenerated Hopf bifurcation point instead of a degenerated homoclinic bifurcation point.

[4] Of course one has to be careful when speaking about the interior of a curve because our phase space is a sphere and not the plane. In the present case we define the interior of the homoclinic orbit as those part of the phase space which contains the angle between the stable and the unstable eigendirection of the saddle point.

Hence the Arnold–Takens–Bogdanov bifurcation has changed its type. In terms of the normal form

$$\dot{x} = y \tag{31}$$
$$\dot{y} = a\,x^2 + b\,x\,y \tag{32}$$

this bifurcation can be attributed to a change in sign of the coefficient b. This degenerated Arnold–Takens–Bogdanov bifurcation is discussed in the mathematical literature on an abstract level [17]. In order to keep the discussion self contained as well as accessible to physicists we have developed a much more elementary approach which nevertheless yields the complete bifurcation diagram of this codimension three bifurcation point. It is given in the appendix.

An additional bifurcation of (at least) codimension three occurs at $h_\perp = 1$. If we approach this point the saddle node bifurcation lines collide at $\gamma = 0, \Delta = 0$ and generate two new cusp points C_3 which initially coincide (cf. Fig.11). The center manifold of this bifurcation is one dimensional because only a single eigenvalue vanishes. On a further increase of the driving field the cusp points C_3 separate. Fig.12 shows the bifurcation diagram.

<Fig.11

<Fig.12

4 Conclusion

Our analysis of a driven spin system has covered both aspects of the dynamics, the Hamiltonian as well as the dissipative case. The undamped linearly polarized driven system discussed in section 2 has shown the typical behaviour of a nonintegrable Hamiltonian system. Our analysis has revealed that only a finite number of resonances occurs in each order of the perturbation expansion. This is a consequence of the direct coupling of the driving field to the action angle variables (cf. eq. (3)) which are in a certain sense the natural variables of the dynamical system. In this respect our system differs from e.g. mechanical oscillators where higher harmonics come into play even at low order resonances. The global perturbative approach is in good agreement with numerical simulations as long as resonace overlapp does not destroy a significant portion of the KAM tori.

The investigation of the dissipative system subjected to a circularly polarized driving field has revealed a surprisingly rich bifurcation scenario. Beside local (saddle node and Hopf) and global bifurcations (homoclinic and saddle node bifurcations of limit cycles) of codimension one we have found five different bifurcations of codimension two and even two bifurcations of codimension three. From our analysis, which has been performed to a large extent analytically, we draw the conclusion that our bifurcation diagram is complete. We notice that for a closely related model, i.e. an antiferromagnet made up of two homogeneous sublattices, a partial bifurcation diagram has been established in [18].

It is widely believed and supported by our results that the Hamiltonian and the dissipative dynamics are entirely different. But the limit of weak damping and the emergence of dissipative dynamics from the Hamiltonian phase space structure is poorly understood. The system analysed in this article provides a physical model to study this issue.

Our investigations constitute a necessary prerequisite for a systematic study of spatially inhomogeneous states of the full Landau–Lifshitz–Gilbert equation. A rather complex behaviour even under the very restrictive assumption of homogeneous magnetization was found. We are convinced that the full spatially inhomogeneous equation, including exchange as well as dipolar interaction, provides the correct theoretical description of strongly driven magnetic systems far from equilibrium. Work in this direction is in progress and will be published elsewhere.

Acknowledgements

One of the authors (W.J.) is indebted to the "Deutsche Forschungsgemeinschaft" for support by a "Habilitandenstipendium". This work was performed within a program of the Sonderforschungsbereich 185 Darmstadt–Frankfurt, Germany.

Appendix: The degenerated Arnold–Takens–Bogdanov Bifurcation

We present here an elementary discussion of the degenerated Arnold–Takens–Bogdanov bifurcation. The mathematical background for this codimension three bifurcation was worked out in ref.[17]. Hence we skip the formal proof of the universality of the unfolding but concentrate on the computation of the bifurcation manifolds in the full three dimensional parameter space.

We consider the case that in the well known normal form of the Arnold–Takens–Bogdanov bifurcation (31,32) a degeneracy of the second order terms occurs, which means that the coefficient b vanishes. In order to establish the normal form of this higher order codimension point we start from the general equations of motion

$$\dot{x}_1 = f_1(\mathbf{x}) = x_2 + \sum_{j=3}^{\infty}\sum_{i=0}^{j} a_{ij} x_1^i x_2^{(j-i)} \tag{33}$$

$$\dot{x}_2 = f_2(\mathbf{x}) = x_1^2 + \sum_{j=3}^{\infty}\sum_{i=0}^{j} b_{ij} x_1^i x_2^{(j-i)} \tag{34}$$

which coincide up to second order with the degenerated situation just described. They contain all possible contributions beyond the second order. In the spirit of normal form calculations we are looking for a coordinate transformation $\mathbf{x} = \mathbf{h}(\mathbf{y})$, $\mathbf{h}(\mathbf{0}) = \mathbf{0}$ including a rescaling of the time $t \to \zeta t$ such that the system (33,34) becomes "as simple as possible" [14]. In the new coordinates we get

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = \zeta \left(D\mathbf{h}(\mathbf{y})\right)^{-1} \mathbf{f}(\mathbf{h}(\mathbf{y})) \quad . \tag{35}$$

Substituting the expansions

$$\begin{aligned} h_1 &= c_{11} y_1 + \sum_{j=2}^{4} \sum_{i=0}^{j} c_{ij} y_1^i y_2^{j-i} \\ h_2 &= d_{11} y_2 + \sum_{j=2}^{4} \sum_{i=0}^{j} d_{ij} y_1^i y_2^{j-i} \end{aligned} \tag{36}$$

into equation (35) and keeping terms up to fourth order we determine the coefficients c_{ij} and d_{ij} in such a way that as many terms as possible vanish in equation (35). The necessary algebraic manipulations are quite extensive. They were performed with the help of a computer program applying symbolic mathematics [19]. We end up with the normal form

$$\begin{aligned} \dot{y}_1 &= y_2 + \mathcal{O}\left(|\mathbf{y}|^5\right) \\ \dot{y}_2 &= y_1^2 + \alpha \;\, y_1^3 + y_1^3 \, y_2 + \mathcal{O}\left(|\mathbf{y}|^5\right) \end{aligned} \tag{37}$$

where α is a constant of order unity which remains undetermined.

In extension and analogy to the two parameter unfoldings of the standard Arnold–Takens–Bogdanov bifurcation [14] we now introduce the following three parameter unfolding

$$\dot{y}_1 = y_2 + \mathcal{O}\left(|\mathbf{y}|^5\right) \tag{38}$$

$$\dot{y}_2 = \mu_1 + \mu_2 \, y_2 + \mu_3 \, y_1 \, y_2 + y_1^2 + \alpha \, y_1^3 + y_1^3 \, y_2 + \mathcal{O}\left(|\mathbf{y}|^5\right) \quad . \tag{39}$$

The bifurcation manifolds in the three dimensional parameter space (μ_1, μ_2, μ_3) will be computed in the sequel.

We begin our discussion with an analysis of the local codimension one bifurcations. Following the lines of section 3.2 we obtain from eqs.(29) easily the saddle node bifurcation manifold

$$\mu_1^{sn} = 0 \quad . \tag{40}$$

Evaluating eqs.(30) a little algebra yields the Hopf bifurcation manifold

$$\mu_2^{\text{Hopf}}(\mu_1, \mu_3) = \sqrt{-\mu_1}\,(\mu_3 - \mu_1) + \mathcal{O}\left(\mu_3 \, \mu_1, \mu_1^2\right) \quad . \tag{41}$$

Fig.13 shows both manifolds in the three dimensional parameter space. < Fig.13

To study the global bifurcations a rescaling is performed which differs from the blowing up used in the treatment of the ordinary Arnold–Takens–Bogdanov bifurcation [14]. One has to choose

$$y_1 = \varepsilon^2\, u, \quad y_2 = \varepsilon^3\, v, \quad \tau = \varepsilon\, t, \tag{42}$$

$$\mu_1 = \varepsilon^4\, \nu_1, \quad \mu_2 = \varepsilon^6\, \nu_2, \quad \mu_3 = \varepsilon^4\, \nu_3 \quad .$$

in order to ensure that the rescaled system becomes Hamiltonian in lowest order and that all non Hamiltonian contributions scale with the same order of magnitude of ε. We obtain from eqs.(38,39)

$$\dot{u} = v + \mathcal{O}\left(\varepsilon^6\right) \tag{43}$$

$$\dot{v} = \nu_1 + u^2 + \varepsilon^2\, \alpha\, u^3 + \varepsilon^5\, \nu_2\, v + \varepsilon^5\, \nu_3\, u\, v + \varepsilon^5\, u^3\, v + \mathcal{O}\left(\varepsilon^6\right) \quad . \tag{44}$$

where the dot now denotes the derivative with respect to the new time τ. We note that the term proportional to ε^2 is Hamiltonian. We are now dealing with a *four* parameter problem where the parameters ν_1, ν_2, ν_3 are of order $\mathcal{O}(1)$ and ε is small. In the limit $\varepsilon \to 0$ eqs.(43,44) yield an integrable Hamiltonian system with the Hamiltonian

$$H(u,v) = \frac{v^2}{2} - \nu_1 u - \frac{u^3}{3} \quad . \tag{45}$$

Fig.14 shows the corresponding phase portrait. It exhibits closed orbits and a homoclinic loop Γ_0 with the energy $H(u,v) = \frac{2}{3}\sqrt{-\nu_1^3}$. < Fig.14

The singular transformation (42) has blown up the degenerated fixed point into a Hamiltonian system because the limit $\varepsilon \to 0$ implies $\mu_1, \mu_2, \mu_3 \to 0$. By perturbing the solutions of the Hamiltonian system (45) we are able to uncover the behaviour of eqs.(43,44) for parameter values μ_1, μ_2, μ_3 close to the origin.

Homoclinic bifurcations occur at those parameter values ν_1, ν_2, ν_3 for which the homoclinic orbit of the Hamiltonian system persists if the perturbation is switched on. This problem can be treated analytically by Melnikovs method (e.g. ref.[14]). The (nondegenerated) zeroes of the time independent Melnikov function[5]

$$\begin{aligned} M(\nu_1,\nu_2,\nu_3) &= \int_{\Gamma_0(\nu_1)} \left(\nu_2\, v + \nu_3\, u\, v + u^3\, v\right)\, du \\ &= \frac{8\sqrt[4]{-4\nu_1^5}}{385}\left(-309\sqrt{-\nu_1^3} + 231\,\nu_2 - 165\,\nu_3\,\sqrt{-\nu_1}\right) \end{aligned} \tag{46}$$

signal the occurrence of the homoclinic bifurcation. In view of the scaling (42) this condition reads

$$\mu_2^{hc} = \frac{103\sqrt{-\mu_1^3} + 55\,\mu_3\,\sqrt{-\mu_1}}{77} \quad . \tag{47}$$

The surfaces at which the Hopf, saddle node and homoclinic bifurcations take place are depicted in Fig.15. < Fig.15

Finally we show that a further global bifurcation, a saddle node bifurcation of limit cycles, comes into existence for the degenerated Arnold–Takens–Bogdanov bifurcation. The subharmonic Melnikov function

$$M_s(\nu_1, \nu_2, \nu_3, E) = \int_{\Gamma(E,\nu_1)} \left(\nu_2\, v + \nu_3\, u\, v + u^3\, v\right)\, du \tag{48}$$

provides an appropriate tool for studying this issue [14]. Here $\Gamma(E, \nu_1)$ denotes a trajectory of the Hamiltonian system (45) with energy E. If this function has a nondegenerated zero for a certain value of the energy then the corresponding periodic orbit persists in the perturbed system (43,44). Moreover a doubly degenerated zero of M_s indicates a saddle node bifurcation of limit cycles at the corresponding parameter value. Hence the saddle node bifurcation manifold is determined by the equations

$$0 = M_s(\nu_1, \nu_2, \nu_3, E) \;=\; \sqrt{2} \int_a^b \frac{(\nu_2\, u + \nu_3 u^2/2 + u^4/4)\,(u^2 + \nu_1)}{\sqrt{E + \nu_1 u + u^3/3}}\, du \tag{49}$$

$$0 = \frac{\partial}{\partial E} M_s(\nu_1, \nu_2, \nu_3, E) \;=\; \sqrt{2} \int_a^b \frac{\nu_2 + \nu_3\, u + u^3}{\sqrt{E + \nu_1 u + u^3/3}}\, du \quad . \tag{50}$$

Here the limits of the integrals are given by the first two zeroes of the argument of the square root:

$$E + \nu_1 u + \frac{u^3}{3} = \frac{1}{3}(u - a)(b - u)(c - u) \qquad a < b < c \quad . \tag{51}$$

We solve the linear system (49,50) for the two parameters ν_2 and ν_3 and obtain a parametric representation of the saddle node bifurcation surface of the form $(\nu_2, \nu_3) = \mathbf{F}(\nu_1, E)$.

Fig.16 contains all bifurcation manifolds in a single diagram. We draw attention to the fact that the saddle node bifurcation manifold for limit cycles meets the Hopf surface in a line of degenerated (local) Hopf bifurcations, whereas its contact with the homoclinic surface yields a line of degenerated (global) homoclinic bifurcations. Both lines represent codimension two bifurcation manifolds. Joyal [16] describes a related phenomenon in a different context. In addition we recall that the bifurcation scenario described at the end of section 3.3 is recovered if one considers two dimensional cross sections (e.g. at constant μ_3) of our three dimensional bifurcation diagram. In closing we restate that the bifurcation set presented in Fig.16 yields the complete bifurcation diagram of the degenerated Arnold–Takens–Bogdanov bifurcation. < Fig.16

[5] By a trivial extension of Melnikov theory the Hamiltonian contribution of second order can be incorporated in eq.(45). Hence it drops from the Melnikov function (46) which is the lowest order of a perturbation series.

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Figure captions

Fig.1 Level lines of the approximate second order invariant f at parameter values $\varepsilon h_\perp = 0.01$, $a = 1$, $H_z = 0.5$, $\omega = 0.2$ and at time $t = 0$.

Fig.2 Level lines of the approximate second order invariant f at parameter values $\varepsilon h_\perp = 0.025$, $a = 1$, $H_z = 0.5$, $\omega = 0.2$ and at time $t = 0$.

Fig.3 Poincaré map at time $t = 0$ of the full Hamiltonian system (3) at parameter values $\varepsilon h_\perp = 0.01$, $a = 1$, $H_z = 0.5$ and $\omega = 0.2$ (cf. Fig.1).

Fig.4 Poincaré map at time $t = 0$ of the full Hamiltonian system (3) at parameter values $\varepsilon h_\perp = 0.025$, $a = 1$, $H_z = 0.5$ and $\omega = 0.2$ (cf. Fig.2).

Fig.5 Bifurcation diagram of eq.(19) at parameter values $h_\perp = 0.1$ and $\Gamma = 0.1$.

The bifurcation lines are denoted as follows: sn: saddle node bifurcation, Ho: supercritical Hopf bifurcation, ho: subcritical Hopf bifurcation, hc: homoclinic bifurcation, and sl: saddle node bifurcation for limit cycles.

Codimension two points are labeled by capital letters: A_i: Arnold–Takens–Bogdanov bifurcation, B: degenerated homoclinic bifurcation, C_i: Cusp bifurcation, H: degenerated Hopf bifurcation, and S_i: saddle node connection.

The numbers refer to the regions described in the text. The region $(\Delta, \gamma) \in [-0.7, -0.69] \times [0.39, 0.395]$ is magnified in the small insert. Note that to the left hand side of the point B there exists a line of a homoclinic bifurcations ending up in point S_2 and a line of a saddle node bifurcation of limit cycles ending up in point H.

Fig.6 Phase portraits in stereographic projection for several points in the bifurcation diagram Fig.5. The numbers refer to the regions in the bifurcation diagram. The phase portraits show actual solutions of the differential equations. Parameter values (Δ, γ) are as follows: (II) $(-0.1, 0.1)$, (V) $(0.75, 0.5)$, (VI) $(-0.1, 0.6)$, (VII) $(0.5, 0.9)$, (VIII) $(-0.65, 0.6)$, and (IX) $(-0.65, 1)$. In all cases there exists an additional fixed point on the lower hemisphere which is not displayed.

Fig.7 Phase portraits at the two different homoclinic lines (a) $\overline{A_1 S_1}$ and (b) $\overline{A_2 S_2}$ (cf. Fig.5). Parameter values are chosen as (a) $\Delta = -0.6$, $\gamma = 0.604$ and (b) $\Delta = -0.4$, $\gamma = 0.4009$. An additional fixed point exists on the lower hemisphere which is not displayed.

Fig.8 Bifurcation diagram of eq.(19) at parameter values $h_\perp = 0.2$ and $\Gamma = 0.1$. The notation is the same as in Fig.5.

Fig.9 Bifurcation diagram of eq.(19) at parameter values $h_\perp = 0.5$ and $\Gamma = 0.1$. The notation is the same as in Fig.5.

Fig.10 Bifurcation diagram of eq.(19) at parameter values $h_\perp = 0.8$ and $\Gamma = 0.1$. The notation is the same as in Fig.5. The lower part shows a magnification of the region near the codimension two point A_2.

Fig.11 Bifurcation diagram of eq.(19) at parameter values $h_\perp = 1$ and $\Gamma = 0.1$. The notation is the same as in Fig.5. The insert shows a magnification of the region $(\Delta, \gamma) \in [-0.02, 0.02] \times [0.2, 0.4]$

Fig.12 Bifurcation diagram of eq.(19) at parameter values $h_\perp = 1.1$ and $\Gamma = 0.1$. The notation is the same as in Fig.5. The insert shows a magnification of the region $(\Delta, \gamma) \in [-0.0015, 0.0015] \times [0.415, 0.445]$

Fig.13 Partial bifurcation set of system (38,39) containing Hopf and saddle node (sn) bifurcation manifolds.

Fig.14 The phase portrait of the Hamiltonian (45) at $\nu_1 = -1$. Γ_0 denotes the homoclinic loop.

Fig.15 Partial bifurcation set of system (38,39) containing the Hopf, saddle node (sn) and homoclinic (hc) bifurcation manifolds (cf. Fig.13).

Fig.16 Complete bifurcation set of system(38,39) containing the Hopf, saddle node (sn) and homoclinic (hc) bifurcation manifolds as well as the saddle node bifurcation manifold of limit cycles (sl) (cf. Fig.15).