On correlation decay in low-dimensional systems

Julia Slipantschuk[(a)], Oscar F. Bandtlow[(b)] and Wolfram Just[(c)]

Queen Mary University of London, School of Mathematical Sciences, Mile End Road, London E1 4NS, UK

PACS 05.45.Ac – Low-dimensional chaos
PACS 02.30.Tb – Operator theory

Abstract – While the decay of correlations in dynamical systems has been discussed in the physics and mathematics literature for several decades, there exists virtually no nontrivial example where the actual decay rates can be computed explicitly. We construct a class of simple dynamical systems for which all correlation properties, in particular, the entire spectrum of the Perron-Frobenius operator, are accessible by analytical means. As an application we discuss an example of an interval map with a small number of branches where the decay of correlations can be made arbitrarily fast. The example can be viewed as the simplest toy model which gives rise to a crossover from transient to asymptotic behaviour with predictable crossover time scale. The model class introduced here points towards relations between correlation decay and Lyapunov spectra.

Introduction. – The decay of correlations in dynamical systems, or to put it in experimentalists' terms, the lineshape in spectra, is one of the most fundamental topics in nonequilibrium statistical physics. The study of mechanisms which cause correlation decay and relaxation processes, i.e., the emergence of dissipative behaviour, has a long tradition in theoretical physics with significant impact for applications. Historically, the question of the emergence of irreversibility and the computation of relaxation rates can be attributed to the celebrated idea of fluctuation dissipation relations as first emphasised by Einstein [1] and then extended by Onsager to the regime close to equilibrium [2]. There have been numerous, mostly not entirely successful attempts to extend these concepts beyond the linear regime, aiming to establish how the underlying microscopic Hamiltonian structure impacts at the macroscopic irreversible level. Recent approaches focus on fluctuation relations (see [3] for a recent overview), but compelling experimental support for such concepts is still missing.

Understanding the occurrence of irreversibility in macroscopic systems based on first principles, i.e., starting from a microscopic level, remains a largely open problem. Simple mathematical toy models play an important role, to uncover the underlying formal structures which cause irreversibility [4]. Even abstract spectral theory is able to contribute to some of these issues. Such ideas are at the heart of the link between statistical physics and modern approaches in dynamical systems theory [5, 6], where both areas mutually benefit. Simple mathematical models like symplectic maps or hyperbolic diffeomorphisms have been proposed as physically relevant classes of systems for which correlation decay can be put on a rigorous basis [7]. Despite the popularity of these approaches to compute resonances and actual values for diffusion coefficients in chaotic Hamiltonian systems, it seems to be virtually impossible to avoid ad hoc approximations, which are quite popular in theoretical physics, but almost impossible to control from a rigorous perspective. In fact, it is quite challenging and technically very demanding to put a spectral theory on a sound basis, see, e.g., [8] and references therein. Thus it is not surprising, that at least to the best of our knowledge, not a single nontrivial model system exists for which resonances can be computed analytically in closed form.

The aim of this article is to make a small contribution in this direction, by explicitly computing the actual values of the relaxation rates in the most basic dynamical models, and illustrating the occurring challenges. For that purpose we will construct a model with arbitrarily fast decay rates, keeping other relevant dynamical quantities almost unaffected.

Low dimensional maps facilitate studying the impact of macroscopic observations on correlation decay, and the

(a) E-mail: `j.slipantschuk@qmul.ac.uk`
(b) E-mail: `o.bandtlow@qmul.ac.uk`
(c) E-mail: `w.just@qmul.ac.uk`

link between decay rates and other dynamical quantities in some detail. Prominent examples in this context are models for anomalous dynamics, which are often based on the impact of marginally stable sets on the resulting algebraic correlation decay [9]. This part of mathematics, for which the term infinite ergodic theory has been coined and which requires the application of subtle techniques, is a very active field of research with considerable impact for applications in physics, see, e.g., [10] for a recent contribution. Here we are concerned with a simpler setup of mixing dynamics, where correlation decay is always exponential and the corresponding transfer operators possess a spectral gap. It is impossible to give a fair account of the entire body of the mathematical literature but a substantial part of the area probably started with [11]. The point we want to stress is that it is surprisingly difficult to estimate the actual exponential decay rate from the properties of the dynamical system, e.g., from the shape of a one dimensional map.

It seems to be a common perception that the time scale of correlation decay in chaotic dynamical systems is related to sensitivity of the underlying dynamics, i.e., with Lyapunov exponents. Even though such relations are not as straightforward as suggested by the analysis of piecewise linear models [12], correlation decay times are often viewed as estimates for Lyapunov exponents [13]. One should, of course, keep in mind that on formal mathematical grounds no relation has to be expected as both time scales, the decorrelation time and the Lyapunov time, probe different formal properties of the underlying transfer operator. The correlation decay is determined by the spectral gap. In contrast, the Lyapunov exponent, if written as an average of the local expansion rate with respect to the invariant measure, can be expressed for our class of systems as the derivative of the so called topological pressure, and therefore is related to the parameter dependence of the largest eigenvalue of the transfer operator. To summarise, the cause of correlation decay and the underlying mechanisms which determine the actual time scales are quite blurred.

It is in fact difficult to estimate numerical values for relaxation rates from the shape of the map, and that task is not facilitated by the absence of any non-trivial example which can be solved by analytical means. Hence we will develop a new paradigmatic model class which gives analytical insight into complex behaviour, in particular, with regards to relaxation properties. In this context we will also address the question whether one can achieve an arbitrarily fast correlation decay in interval maps.

A rigorous account. – We consider analytic expanding full branch interval maps $T : [-1,1] \to [-1,1]$, i.e., maps which consist of analytic branches with slope larger than one such that each branch maps onto the entire interval $I = [-1,1]$ (see, e.g., Fig. 1). Furthermore, we require that the map can be extended to an analytic map on the complex unit circle $\tau\colon S^1 \to S^1$, by $\tau(\exp(i\pi x)) = \exp(i\pi T(x))$; that is, the different branches match up in an analytic way. Maps of this type are known to have nice dynamical properties. In particular, every such map possesses an invariant measure μ which is given in terms of an analytic density, and the correlation function for analytic observables φ and ψ

$$C_{\varphi,\psi}(n) = \int (\varphi \circ T^n)\psi \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu \int \psi \, \mathrm{d}\mu, \quad (1)$$

decays exponentially. The rate of decay is determined by the spectral properties of the Perron-Frobenius operator associated to T. If $\Phi_1, \ldots, \Phi_K$ are the inverse branches of the map T, we can choose a neighbourhood $D \subset \mathbb{C}$ of I such that the associated Perron-Frobenius operator

$$\mathcal{L}_I f = \sum_{k=1}^{K} \Phi_k' \cdot (f \circ \Phi_k) \quad (2)$$

is well defined and compact when considered on the Banach space of bounded holomorphic functions on D equipped with the supremum norm [14].

It is well established but probably not widely known that the spectral properties of the Perron-Frobenius operator $\mathcal{L}_I$ are linked with the properties of the complex analytic map τ on the complex unit circle (see, e.g., the remark in [15]). Eigenvalues and eigenfunctions of $\mathcal{L}_I$ fall into two classes [16]. Eigenfunctions which are periodic correspond to eigenfunctions of the Perron-Frobenuis operator $\mathcal{L}_{S^1}$ of the associated map τ on the complex unit circle, and vice versa. In addition, there are eigenvalues which are given by the inverse powers of the slope of T at the fixed point at the interval endpoint, $(T'(-1))^{-n}$, $n \geq 1$. The corresponding eigenfunctions cannot be extended analytically to the complex unit circle. Hence, solving the eigenvalue problem for the operator $\mathcal{L}_I$ reduces to solving the corresponding problem of $\mathcal{L}_{S^1}$ and using the following relation between their spectra

$$\sigma(\mathcal{L}_I) = \sigma(\mathcal{L}_{S^1}) \cup \{(T'(-1))^{-n} : n \geq 1\}. \quad (3)$$

The simplest example illustrating this is the well-known Bernoulli map on the interval $T(x) = 2x - \mathrm{sgn}(x)$, whose Perron-Frobenius operator has eigenvalues $1/2^n$, $n \geq 0$, which are caused by the fixed point slope. On the complex unit circle this maps translates to $\tau(z) = z^2$. This dynamical system only has the trivial spectrum $\{0, 1\}$, i.e., correlations for periodic analytic observables decay super-exponentially.

This dichotomy naturally raises the question whether there can be a relation between mixing rates and other measures of chaoticity such as entropy or Lyapunov exponents. Partial answers can be found, e.g., as upper bounds of the mixing rate in terms of the entropy for real analytic suspension flows [17], and in terms of Lyapunov exponents for piecewise linear Markov maps [18]. One purpose of this note is to show that there are expanding interval maps for which the exponential rate of decay can be made arbitrarily large whereas the Lyapunov exponent or the entropy remains bounded.

To address such an issue one needs a sufficiently rich class of models where the entire spectrum of the Perron-Frobenius operator is accessible. It is well known that analytic maps acting on the complex unit circle which are also analytic in the entire unit disk can be written as so-called finite Blaschke products [19]

$$\beta(z) = \exp(i\gamma) \prod_{k=1}^{K} \frac{z - a_k}{1 - z\bar{a}_k}, \quad (|a_k| < 1, K > 1) . \quad (4)$$

These induce full branch interval maps if the complex phase is chosen such that $\beta(-1) = -1$, as each factor has modulus one on the unit circle. Expansivity of the map is equivalent to β having a unique fixed point $z_* = \beta(z_*)$ within the complex unit disk. Such a fixed point then determines the invariant density of the interval map via the analytic expression

$$\rho(x) = (1 - |z_*|^2)/(2|\exp(i\pi x) - z_*|^2) . \quad (5)$$

Apart from these classical results it is a recent discovery [20] that this fixed point also determines the entire spectrum of the Perron-Frobenius operator associated to the corresponding map on the unit circle:

$$\sigma(\mathcal{L}_{S^1}) = \{1, 0\} \cup \{(\beta'(z_*))^n, \left(\overline{\beta'(z_*)}\right)^n : n \geq 1\} . \quad (6)$$

Using eq.(3) all eigenvalues of the Perron-Frobenius operator on the interval, $\mathcal{L}_I$, are then easily computed by supplementing eq.(6) with the powers of the inverse slope at the interval endpoint. Finite Blaschke products thus constitute an ideal laboratory for testing certain aspects of dynamical systems theory as all relevant quantities are accessible.

Decay of autocorrelations. – We shall now construct a dynamical system with arbitrarily fast correlation decay, but with the counter-intuitive constraint of leaving other dynamical quantities such as the Lyapunov exponent and the invariant density essentially unaffected. Following the previous section, the idea is to construct a Bernoulli-type map but with large fixed point slope. To this end we consider the finite Blaschke product

$$\beta(z) = z^2 \frac{z - b}{1 - bz} \quad (7)$$

for $b \in (-1, 1)$. It is straightforward to compute that eq.(7) induces a three-to-one analytic expanding map

$$B(x) = 3x + \frac{2}{\pi} \arctan\left(\frac{b \sin(\pi x)}{1 - b\cos(\pi x)}\right) - 2m \quad (8)$$

on the interval $I = [-1, 1]$. Here m labels the three branches of the map, where $m = -1$ if $x < -\arccos((1 + b)/2)/\pi$, $m = 1$ if $x > \arccos((1 + b)/2)/\pi$, and $m = 0$ otherwise (see Fig. 1). The map B has fixed points at -1 and 1 with slope $B'(-1) = B'(1) = (3 + b)/(1 + b)$. For the trivial case $b = 0$, eq.(8) yields the tripling map, while a non-zero parameter value induces curvature. In the pointwise limit $b \to -1$, eq.(8) approaches a version of the Bernoulli map with offset. This limiting case will play an important role in our investigation.

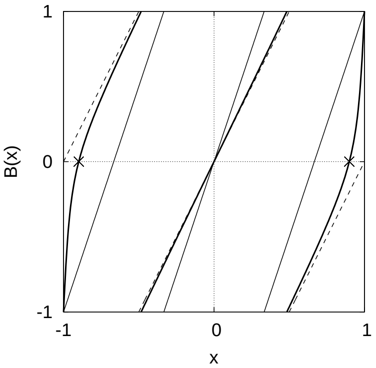

Fig. 1: The interval map in eq.(8) for parameter values $b = 0$ (thin solid) and $b = -0.95$ (thick solid). The dashed line shows the pointwise limit as $b \to -1$. The two crosses indicate the non-trivial zeros of the map for $b = -0.95$.

The fixed point of eq.(7) within the unit disk is given by $z_* = 0$ for any value of the parameter b. Thus, the invariant density eq.(5) of the interval map eq.(8) is constant. Hence, for any $b \in (-1, 1)$ the Lyapunov exponent for B with respect to this invariant measure is given by $\Lambda = \int_I \ln B'(x)\mathrm{d}x/2$ and with little effort can be calculated explicitly as

$$\Lambda = \ln\left(\left(3 + b^2 + \sqrt{(1 - b^2)(9 - b^2)}\right)/2\right) , \quad (9)$$

see Fig. 2. In addition, we have $\beta'(z_*) = 0$, and by eq.(6) there is no contribution from periodic eigenfunctions to the point spectrum of the Perron-Frobenius operator $\mathcal{L}_I$ in eq.(2). Thus all eigenvalues are caused by the inverse of the fixed point slope at $x = -1$ and we end up with

$$\sigma(\mathcal{L}_I) = \{1, 0\} \cup \{((1 + b)/(3 + b))^n : n \geq 1\} . \quad (10)$$

The exponential rate of decay α for the correlation function in eq.(1) for generic analytic observables φ and ψ is related to the subleading eigenvalue $\mu_2 = (1 + b)/(3 + b)$ of $\mathcal{L}_I$ via $\alpha = -\ln \mu_2 = -\ln((1 + b)/(3 + b))$. Obviously, all the eigenvalues can be made arbitrarily small and the correlation decay arbitrarily fast, if the parameter b is chosen close to -1, while the Lyapunov exponent Λ in eq.(9) and the invariant density remain largely unaffected (see Fig. 2). Hence, α is not bounded by any finite multiple of Λ as b approaches -1, unlike the case of piecewise linear Markov maps considered in [18], where a bound $\alpha \leq 2\Lambda$ has been established.

How are the spectral properties discussed so far reflected in the actual shape of the correlation function? On the one

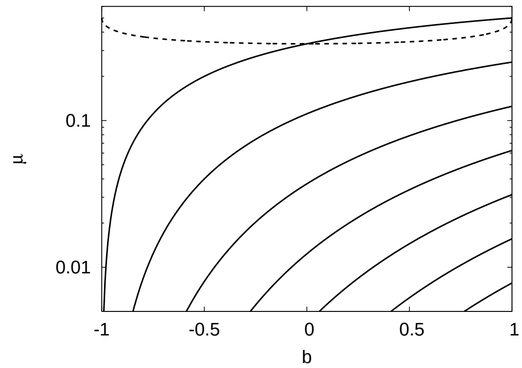

Fig. 2: Eigenvalues of the Perron-Frobenius operator in eq.(10) for the interval map in eq.(8), as a function of the parameter b (solid lines), in logarithmic scale. The dashed line shows the inverse Lyapunov multiplier $\exp(-\Lambda)$, see eq.(9).

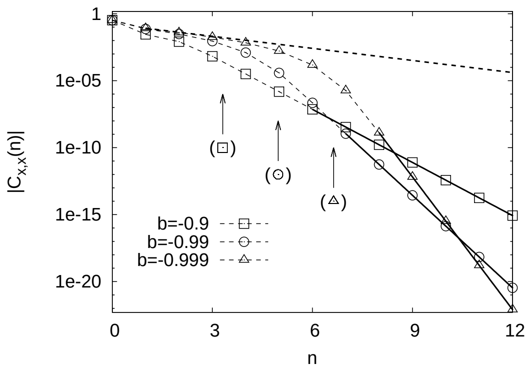

Fig. 3: Autocorrelation function eq.(11) of the map B in eq.(8) for different values of the parameter b (symbols with dashed lines as guide for the eye), on a logarithmic scale. Solid straight lines indicate the asymptotic decay as computed from the subleading eigenvalue of the Perron-Frobenius operator. The dashed straight line displays the correlation function $C_{x,x}(n) = -1/(6 \times 2^n)$ of the pointwise limit map as $b \to -1$ (see Fig. 1). Arrows show the respective estimates of the crossover time scale n_c according to eq.(12).

hand, as all eigenvalues approach zero in the limit $b \to -1$ one expects a very fast asymptotic decay. On the other hand, the actual map B largely looks like a shifted version of the Bernoulli map (see Fig. 1) and one would naively expect an exponential decay according to the inverse slope. To clarify the picture let us look in some detail at the coordinate autocorrelation function, i.e., at eq.(1) for the observables $\phi(x) = \psi(x) = x$. As the mean value vanishes we are concerned with evaluating the integrals

$$C_{x,x}(n) = \frac{1}{2}\int_I x B^{(n)}(x)\,\mathrm{d}x\,. \tag{11}$$

It is fairly straightforward but slightly tedious to work out these integrals numerically to high precision. Given n, it is convenient to split the range of integration I into certain small intervals called cylinder sets, on which the kernel $xB^{(n)}(x)$ is monotonic. Each of these integrals is then computed with a suitable integration routine. For that purpose we have used a quadruple precision version of the QUADPACK routines [21]; the result is displayed in Fig. 3. The asymptotic decay of the correlation function is determined by the subleading eigenvalue. However, as the parameter b approaches -1 the autocorrelation function develops a pronounced transient exponential shape which follows the correlation decay of the shifted Bernoulli map. In this way the correlation function bridges the dichotomy pointed out in the previous paragraph.

There is a crossover between the transient (slow) exponential decay governed by the average slope of the map and the (fast) exponential decay determined by the maximal slope, i.e., by the subleading eigenvalue of the Perron-Frobenius operator. One can develop a simple heuristic argument to estimate the time scale at which this transition between the two different exponential regimes takes place. For the correlation function eq.(11) to be affected by the fine structure of the map B, the time n has to be sufficiently large, allowing the dynamics to explore the small regions near $x = \pm 1$ where the map has large slope. To quantify this argument we estimate on the one hand the size Δ of these two intervals by the zeros of the map close to $x = \pm 1$ (see Fig. 1). These zeros follow either from eq.(7) or eq.(8), yielding $\pm \arccos((b-1)/2)/\pi$ so that $\Delta = 1 - \arccos((b-1)/2)/\pi \simeq \sqrt{1+b}/\pi$ as $b \to -1$. On the other hand, superficially the map B looks like the shifted Bernoulli map if b is close to -1. Hence, for small values of n the correlation decay follows that of the Bernoulli map which is given by 2^{-n}. The regions at the end of the interval where the map B substantially differs from the Bernoulli map give a contribution of the order of Δ to the integral in eq.(11) as the kernel is bounded by one. This contribution becomes visible, causing the correlation decay to deviate from that of the Bernoulli map, when Δ and 2^{-n} become comparable.

One can rephrase this argument by partitioning the range of integration into cylinder sets, used previously in the actual numerical integration. In the limit $b \to -1$ the size of most of these intervals can be estimated by 2^{-n}. Apart from intervals close to the endpoints $x = \pm 1$ the kernel essentially looks like the one obtained from the shifted Bernoulli map, resulting in the decay visible for small values of n. However, the correlation function starts to deviate for n sufficiently large, when the size 2^{-n} of the intervals close to the boundary becomes comparable to the size Δ of the exceptional region. In short, from the dynamical perspective the correlation integral in eq.(11) at time n resolves phase space features which are averaged over cylinder sets of the corresponding generation. Hence, the balance condition $2^{-n} \simeq \Delta \simeq \sqrt{1+b}/\pi$ gives an estimate for the crossover time scale which results in

$$n_c \simeq -\ln(1+b)/(2\ln 2), \quad (b \to -1)\,. \tag{12}$$

This simple argument predicts rather well the actual structure visible in Fig. 3. The formal challenge of course remains to develop a consistent and general mathematical framework for evaluating such features based on spectral properties of the Perron-Frobenius operator.

Conclusion. – Given a dynamical system, the decay rate of correlations is difficult to predict, using simple features of the system only. The absence of analytically solvable examples which can be investigated in detail has hindered further progress in this direction in the past. The family of examples presented here constitutes a new paradigm and makes progress in this direction possible. It can be easily extended to account for more sophisticated features such as coupled systems and spatial degrees of freedom, thus shedding light on correlation decay in coupled complex structures.

Our example shows that even in the simplest possible setup, correlation decay can be made arbitrarily fast, independently of the Lyapunov exponent. Such an observation is somewhat counter-intuitive and shows that one cannot expect correlation decay to be caused by a simple mechanism. In particular, a strongly unstable fixed point on its own is by no means a sufficient condition for the fast decay reported here. As the analysis proves, a delicate global balance of dynamical features, which, in the present case, is induced by the required analytical properties, is essential. Nevertheless, expansivity, but not the Lyapunov exponent, is still one of the key ingredients for the decay of correlations. It is easy to confirm that for interval maps which can be extended analytically to the circle, the maximal expansion rate

$$\Lambda_+ = \lim_{n\to\infty} \sup \left\{ \frac{1}{n} \ln |(T^n)'(x)| : x \in I \right\} \qquad (13)$$

determines a lower bound for the subleading eigenvalue via $|\mu_2| \geq \exp(-\Lambda_+)$, or equivalently an upper bound for the mixing rate, $\alpha \leq \Lambda_+$. This rigorous statement is in line with previously reported relations between mixing rates and generalised Lyapunov spectra [12].

We have designed our model systems in such a way that a rigorous spectral theory can be developed in an analytic setup. This requires the maps to be structurally stable with exponential correlation decay, thus excluding the treatment of bifurcations. Such constraints may raise the question whether our findings can be considered to be typical and relevant in an applied setup. In order to partially answer such a question, let us consider the implications if transitions and bifurcations can occur. One simple model for this is the Smale incomplete tent map, where a well-defined spectral theory is still available (see, e.g., [22] for an accessible exposition). Due to structural instability, analytic features are destroyed and highly discontinuous properties become visible and mathematically relevant. The crucial question is then which observables are physically relevant, and on which function space the Perron-Frobenius operator should be meaningfully established. Nice properties such as compactness of the operator cannot generally be taken for granted, and standard perturbation schemes used frequently in physics do not apply without caveats. All these issues affect the question of correlation decay (see, e.g., [18] for a recent discussion). In the context of a phenomenological modelling approach of macroscopic behaviour a smooth continuous description may be the method of choice. In this setup our models would be of the preferred class. The archetype for this approach is the seminal study of the applicability of standard linear response theory to maps, see [23]. Thus, while there is no proof that the class of models introduced here is generic, we think there is evidence for the applicability of our findings to real world problems.

It should then be discussed whether all the features found here survive in a real world environment, e.g., whether they are masked by noise or imperfections which are prevalent in any experimental setting. In fact, while the spectrum of the Perron-Frobenius operator determines the asymptotic correlation decay, the present explicit example demonstrates that one has to be careful about its actual implications. On an intermediate time scale, probably relevant in real applications, the correlation decay may be governed by different mechanisms. Rigorous mathematical tools to cover such features and the observed crossover still need to be developed; pseudospectra could provide a suitable tool (see e.g. [24] for a classical reference on this subject). A benchmark for success would be to capture all aspects of such a dynamically generated finite time scale phenomenon with the intrinsically generated dynamical crossover towards the asymptotic behaviour.

* * *

The work has been supported by EPSRC through grant No. EP/H04812X/1. WJ gratefully acknowledges support by SFB910 and the kind hospitality by Eckehard Schöll and his group during a stay at TU Berlin.

REFERENCES

[1] Einstein A., *Ann. Phys.*, **322** (1905) 549
[2] Onsager L., *Phys. Rev.*, **37** (1931) 405
[3] Klages R., Just W. and C. Jarzynski C. (Editors), *Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond*, (Wiley-VCH, Weinheim) 2012
[4] Courbage M. and Prigogine I., *PNAS*, **80** (1983) 2412
[5] Ruelle D., *Thermodynamic formalism* (Addison-Wesley) 1978
[6] Katok A. and Hasselblatt B., *Introduction to the Modern Theory of Dynamical Systems* (Camb. Univ. Press) 1997
[7] Ruelle D., *Phys. Rev. Lett.*, **56** (1986) 405
[8] Gouezel S. and Liverani C., *Ergod. Th. & Dyn. Sys.*, **26** (2006) 189
[9] Young L.S., *Israel J. Math.*, **110** (1999) 153
[10] Melbourne I. and Terhesiu D.,

Ergod. Th. & Dyn. Sys., **FirstView** (2013) 1 doi:10.1017/etds.2013.59
[11] Hofbauer F. and Keller G., *Math. Z.*, **180** (1982) 119
[12] Christiansen F., Paladin G and Rugh H.H, *Phys. Rev. Lett.*, **65** (1990) 2087
[13] Sprott J.C., *Chaos and time-series analysis* (Oxford Univ. Press) 2003
[14] Bandtlow O.F and Jenkinson O., *Ergod. Th. & Dyn. Sys.*, **28** (2008) 1701
[15] Keller G. and Rugh H.H, *Nonlinearity*, **17** (2004) 1723
[16] Slipantschuk J., O. F. Bandtlow O.F. and Just W., *Analytic expanding circle maps with explicit spectra (preprint), http://arxiv.org/abs/1306.0445*, (2013)
[17] Naud F., *Ann. Henri Poincaré*, **10** (2009) 429
[18] Slipantschuk J., Bandtlow O.F., and Just W., *J. Phys. A*, **46** (2013) 075101
[19] Martin N.F.G., *Bull. Lond. Math. Soc.*, **15** (1983) 343
[20] Bandtlow O.F., Just W., and Slipantschuk J., *Spectral structure of transfer operators for expanding circle maps (preprint)*, (2013)
[21] *http://www.netlib.org/quadpack/*,
[22] Dörfle M., *J. Stat. Phys.*, **40** (1985) 93
[23] Grossmann S., *Z. Phys. B*, **57** (1984) 77
[24] Trefethen L. N. and Embree M., *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators* (Princeton Univ. Press) 2005