Influence of control loop latency on time–delayed feedback control

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As realised recently the success of delayed feedback control methods may be significantly restricted by control loop latency, i. e. by an additional delay which acts on the control force. We show within a linear stability analysis that such a limitation is caused by the shift of frequency splitting points. Our analytical results are in good quantitative agreement with numerical "exact" calculations of the Toda oscillator and with data from an electronic circuit experiment.

The topic of control has become popular among physicists in the last few years in particular in connection with the stabilization of periodic orbits which are embedded in a chaotic attractor. In that context delayed feedback control methods which are easy to apply in real experimental situations [1,2] have been rediscovered [3]. Meanwhile several features of such control schemes have been understood even analytically. In particular, torsion of neighbouring trajectories is important for the scheme to work at all [4,5], the limitations caused by the length of the period and the size of the Floquet exponents can be relaxed by including integer multiple delays [6,7], and the appropriate delay time can be determined from properties of the control signal [8–10] if the periods are not known a priori (cf. also [11] for recent reviews). In actual experimental realizations of delayed feedback methods the control force is generated electronically. Recent experiments on electronic circuits and numerical simulations have demonstrated that the additional time lag of these devices may strongly limit the success of the control scheme [12]. From the general point of view of control theory such an observation is not quite new and well known by engineers for several decades within the context of stabilising time independent states (cf. e. g. [13] where the influence of physiological delay on the balancing of a stick by a human is studied). Here we emphasise that the discussion of such control loop latencies is of particular importance since delayed feedback methods have been designed for control in fast experimental systems. Latency has turned out to be one of the decisive limitations for successful control. In spite of its practical importance a systematic investigation of this problem has been missing so far. Since latency is inherent in any fast experimental system, it is essential to have estimates for its maximal allowed value and its relation to the Floquet exponent of the uncontrolled orbit. By covering these open questions we consider our investigation to be of great importance for all practical applications of time–delayed feedback control.

Theoretical approach – Following the idea of [4] we consider a general dynamical system. Let $\boldsymbol{x}(t)$ denote the internal degrees of freedom and suppose that a scalar signal $g[\boldsymbol{x}(t)]$ is accessible to measurements. From the latter a control force $g[\boldsymbol{x}(t-\delta)] - g[\boldsymbol{x}(t-\delta-\tau)]$ is generated, where δ denotes the control loop latency. The equation of motion which fits within this setup reads

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}(t), K\{g[\boldsymbol{x}(t-\delta)] - g[\boldsymbol{x}(t-\delta-\tau)]\}) \quad . \quad (1)$$

Here the control amplitude K determines the strength of the feedback. Although we presuppose that the control amplitude acts as a multiplication factor, our considerations can be extended easily to include much more general dependencies without any essential modification. We do not specify the analytic dependence on the control force in order to keep the approach as general as possible. The system without control, $K = 0$, should admit an unstable periodic orbit $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(t+\tau)$ which we intend to stabilize. First of all this orbit is not modified by the control force, since the delay has been fixed according to the period. The influence of the control loop latency on the control scheme is investigated by means of a linear stability analysis. If we take the Floquet theory into account [14], the deviations from the orbit obey $\boldsymbol{x}(t) - \boldsymbol{\xi}(t) \simeq \exp[(\Lambda + i\Omega)t]\boldsymbol{U}(t)$, where eq.(1) yields for the dominant exponent and the eigenvector

$$(\Lambda + i\Omega)\,\boldsymbol{U}(t) + \dot{\boldsymbol{U}}(t) = D_1\boldsymbol{F}(\boldsymbol{\xi}(t),0)\boldsymbol{U}(t) + d_2\boldsymbol{F}(\boldsymbol{\xi}(t),0) \cdot \{Dg[\boldsymbol{\xi}(t)]\boldsymbol{U}(t-\delta)\}\kappa, \quad \boldsymbol{U}(t) = \boldsymbol{U}(t+\tau) \quad . \quad (2)$$

Here the abbreviation

$$\kappa := K\exp[-(\Lambda+i\Omega)\delta]\,\{1-\exp[-(\Lambda+i\Omega)\tau]\} \quad (3)$$

has been introduced. Dg denotes the gradient of $g[\boldsymbol{x}]$, $D_1\boldsymbol{F}$ and $d_2\boldsymbol{F}$ the derivative with respect to the first

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vector type and the second scalar argument, respectively. The real part Λ and the imaginary part Ω, which of course depend on the control amplitude K, determine the stability as well as the torsion of the orbit. The corresponding values without control, $K = 0$, are denoted by small letters $\lambda > 0$ and ω for convenience.

Eq.(2) determines the properties of the control scheme. The right hand side depends on the exponent and the control amplitude through the single parameter (3) only. Let us for the moment consider the latter value to be given. Then, the Floquet exponent of the operator on the right hand side of eq.(2) depends of course on κ and δ, and we denote this quantity by $\Gamma_\delta[\kappa]$. Eq.(2) tells us that the exponent $\Lambda + i\Omega$ which we are looking for coincides with $\Gamma_\delta[\kappa]$, i. e. it obeys the constraint

$$\begin{aligned}&\Lambda + i\Omega \\ &= \Gamma_\delta\left[K \exp[-(\Lambda + i\Omega)\delta]\left\{1 - \exp[-(\Lambda + i\Omega)\tau]\right\}\right] \quad .\end{aligned} \tag{4}$$

By presupposition $\Gamma_\delta[0] = \lambda + i\omega$ holds, since eq.(2) reduces to the uncontrolled dynamics in the case $\kappa = 0$. Employing a Taylor series expansion eq.(4) results to first order in

$$\begin{aligned}\Lambda + i\Omega = \lambda + i\omega + \chi(\delta) K \exp[-(\Lambda + i\Omega)\delta] \\ \cdot \left\{1 - \exp[-(\Lambda + i\Omega)\tau]\right\} + \mathcal{O}(K^2) \quad .\end{aligned} \tag{5}$$

Here the first Taylor coefficient $\chi(\delta)$ can be computed from eq.(2) by a usual perturbation expansion, which is well known for time independent cases e. g. from quantum mechanics. If we introduce the left– and right–Floquet eigenvectors $\boldsymbol{v}^T(t) = \boldsymbol{v}^T(t+\tau)$ and $\boldsymbol{u}(t) = \boldsymbol{u}(t+\tau)$ of the uncontrolled dynamics ($\kappa = 0$) as

$$\begin{aligned}(\Lambda + i\Omega)\,\boldsymbol{u}(t) + \dot{\boldsymbol{u}}(t) &= D_1 \boldsymbol{F}(\boldsymbol{\xi}(t), 0)\boldsymbol{u}(t) \\ (\Lambda + i\Omega)\,\boldsymbol{v}^T(t) - \dot{\boldsymbol{v}}^T(t) &= \boldsymbol{v}^T(t) D_1 \boldsymbol{F}(\boldsymbol{\xi}(t), 0) \quad ,\end{aligned} \tag{6}$$

then the coefficient $\chi(\delta)$ is given by the corresponding matrix element of the coupling matrix

$$\begin{aligned}\chi(\delta) = \int_0^\tau \{\boldsymbol{v}^T(t) d_2 \boldsymbol{F}(\boldsymbol{\xi}(t), 0)\}\{Dg[\boldsymbol{\xi}(t)]\boldsymbol{u}(t-\delta)\}\, dt \\ \Big/ \int_0^\tau \boldsymbol{v}^T(t)\boldsymbol{u}(t) \quad .\end{aligned} \tag{7}$$

The complex number $\chi(\delta)$ is obviously periodic in the delay, $\chi(\delta) = \chi(\delta + \tau)$.

In order to simplify the subsequent considerations we concentrate on unstable periodic orbits which flip their neighbourhood during one turn, i. e. $\omega = \pi/\tau$. Although such a choice may look at the first glance very special, we emphasise that torsion, i.e. a nonvanishing frequency, is a necessary prerequisite for the control to work at all. Furthermore, in three–dimensional dissipative systems only orbits which either perform a complete flip during one turn or orbits without torsion can occur. Hence, our choice covers the case of simple electronic circuit systems. For our particular value of ω, which of course is a structurally stable situation, we can get rid of the complex values of our quantities. In fact, from definition (6) follows that $\hat{\boldsymbol{u}}(t) := \exp(i\pi t/\tau)\boldsymbol{u}(t)$ and $\hat{\boldsymbol{v}}^T(t) := \exp(-i\pi t/\tau)\boldsymbol{v}^T(t)$ are real valued. With these properties expression (7) results in

$$\chi(\delta) = \exp(i\pi\delta/\tau)\rho(\delta), \quad \rho(\delta) = -\rho(\delta + \tau) \tag{8}$$

where ρ is a real quantity and the antiperiodicity follows from the periodicity of χ. If we insert eq.(8) into eq.(5) and neglect the second order contribution we end up with

$$\begin{aligned}\Lambda\tau + i\Delta\Omega\tau = \lambda\tau - (-\tau\rho(\delta))K \exp[-(\Lambda\tau + i\Delta\Omega\tau)\delta/\tau] \\ \cdot \left\{1 + \exp[-\Lambda\tau - i\Delta\Omega\tau]\right\} \quad ,\end{aligned} \tag{9}$$

where $\Delta\Omega = \Omega - \pi/\tau$ denotes the frequency deviation and dimensionless quantities have been introduced. The dependence of the Floquet exponent $\Lambda + i\Omega$ on control amplitude and control loop latency can now be evaluated from eq.(9). The only system dependent and yet undetermined quantity $\rho(\delta)$ just sets the scale for the control amplitude. One should, however, keep in mind that this scaling factor vanishes at some value $0 < \delta < \tau$ because of the antiperiodicity (8). Although in such a case higher order terms will become important one already recognizes a mechanism by which the efficiency of the control scheme is suppressed.

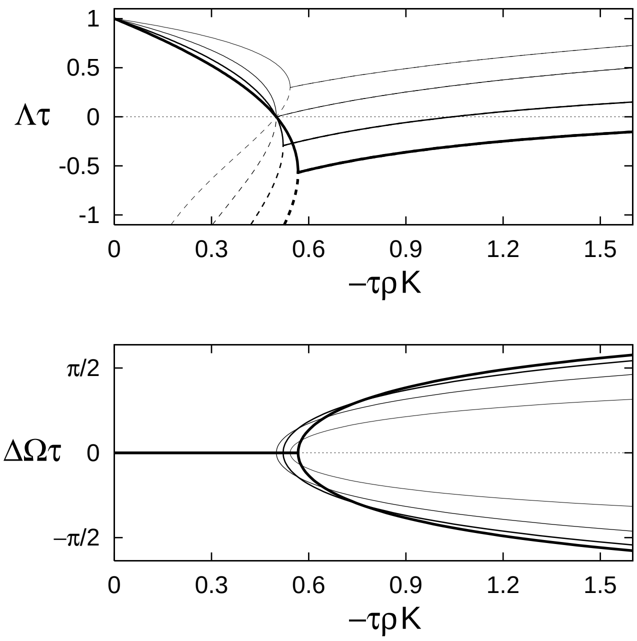

FIG. 1. Real part of the Floquet exponent and frequency deviation in dependence on the rescaled control amplitude, obtained from the analytical expression (9) for $\lambda\tau = 1$ and different values of δ: 0, 0.2τ, 0.5τ, and τ (decreasing line thickness corresponds to increasing values of δ). The broken line indicates the corresponding second real Floquet multiplier.

The dependence of the exponent on the rescaled control amplitude for different values of the control loop latency is summarised in Fig.1. On increasing the control amplitude the exponent Λ decreases and eventually changes its sign at $(-\tau\rho)K_{min} = \lambda\tau/2$. Hence the orbit becomes stable in a flip bifurcation. On further increase of $(-\tau\rho)K$ two exponents collide giving rise to a nonvanishing frequency deviation. Beyond that value the real part increases again and may finally change sign at a control amplitude K_{max}, so that the orbit loses stability via a Hopf bifurcation. As a consequence a finite interval of control amplitudes is obtained, where successful control is possible. For increasing control loop latency the frequency splitting point shifts towards positive values of $\Lambda\tau$ so that the control interval shrinks in size. Beyond a critical value

$$\delta_c = \tau(1 - \lambda\tau/2)/(\lambda\tau) \qquad (10)$$

the frequency splitting occurs for positive values of $\Lambda\tau$ so that stabilization is no longer achieved.

Numerical analysis – Within our theoretical approach we have adopted a single approximation, i. e. a first–order Taylor series truncation, to estimate the influence of the control loop latency. Although one cannot expect that such an approximation always yields quantitatively correct predictions, one may ask whether the qualitative features are reproduced correctly. To address this question we resort to a numerical analysis of the driven and damped Toda oscillator subjected to delayed feedback control

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_2 - \alpha[\exp(x_1) - 1] + A\sin(2\pi t) \\ &\quad - K\left\{x_2(t-\delta) - x_2(t-\delta-\tau)\right\} \quad . \end{aligned} \qquad (11)$$

The parameters of the oscillator are set to the values $\mu = 0.8$, $\alpha = 25$, and $A = 105$ to ensure a chaotic uncontrolled dynamics, and we focus on the unstable period–one orbit with Floquet exponent $\lambda = 1.391\ldots$ and $\omega = \pi$. Within the notation of the preceding paragraph the control force is derived from the scalar signal $g[\boldsymbol{x}] = x_2$ and the delay is adjusted to the period, $\tau = 1$.

In order to check the accuracy of our previous expansion we focus on a numerical analysis of the stability problem for this unstable orbit (cf. eq. (2)). Although the reduction to an ordinary Floquet problem, which is very fruitful in the case without control loop latency (cf. [15]) does not apply, the eigenvalue with largest real part can be found by a plain integration of the linearised equation. The dependence of the Floquet exponent on the control amplitude is summarised in Fig.2 for different values of the control loop latency. One observes essentially the same features that have been predicted theoretically. In particular, the flip and Hopf bifurcations giving rise to the finite control interval seem to be a quite general feature of the control method. One typically observes a whole period doubling sequence below K_{min} through which the control signal becomes chaotic if the control amplitude is lowered. We stress that the size of the control interval shrinks on increasing δ.

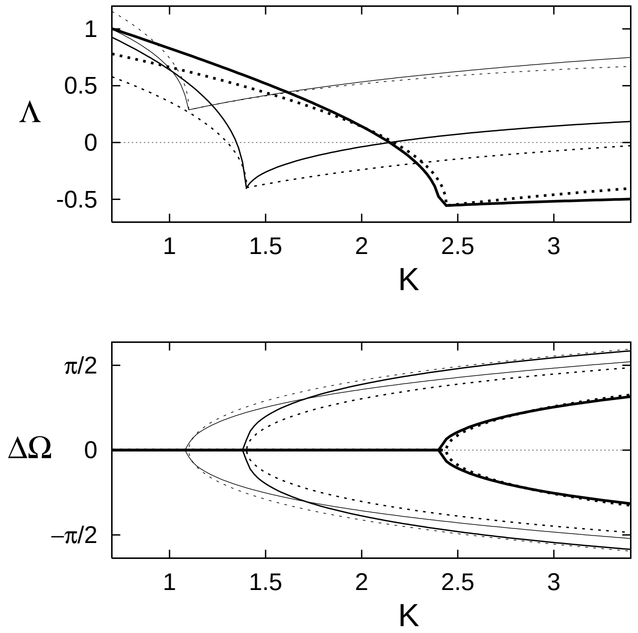

FIG. 2. Real part of the Floquet exponent and frequency deviation in dependence on the control amplitude, obtained for the Toda oscillator (11) and different values of δ: 0, 0.15, and 0.3 (solid lines, decreasing line thickness corresponds to increasing values of δ). The dotted lines indicate the corresponding analytical result (9) with the fit parameters $[\lambda\tau; -\tau\rho] = [1.02; 0.24]$ ($\delta = 0$), $[0.94; 0.36]$ ($\delta = 0.15$), and $[1.66; 0.78]$ ($\delta = 0.3$).

Furthermore, we compare the numerical "exact" Floquet exponent with the analytical expression (9). The latter has been obtained formally as a first–order Taylor series expansion in the control amplitude. However, if one recalls the derivation there is no need to perform the expansion at $K = 0$. One can expand eq.(4) at any real value of the argument. The price one has to pay is that the term of order zero, $\lambda\tau$, is not given any longer by the Lyapunov exponent of the uncontrolled orbit, but may be viewed as an additional fit parameter. Here we adopt such a point of view which is quite reasonable since in applications the exponent of the uncontrolled orbit is often unknown. We fix δ/τ to its known numerical value but adjust $\lambda\tau$ and $(-\tau\rho(\delta))$ in eq.(9) in such a way that the frequency splitting point is properly reproduced. If one recalls that the analytical expression is just a first–order series expansion, the coincidence is quite satisfactory even from a quantitative point of view. Altogether, our analysis shows that the theoretical approach captures the essential features of the control loop latency.

Experiments – To illustrate the consequences of our analytical results for experiments we have performed measurements on an autonomous nonlinear electronic circuit. A diagram of the experimental set–up is displayed in Fig.3.

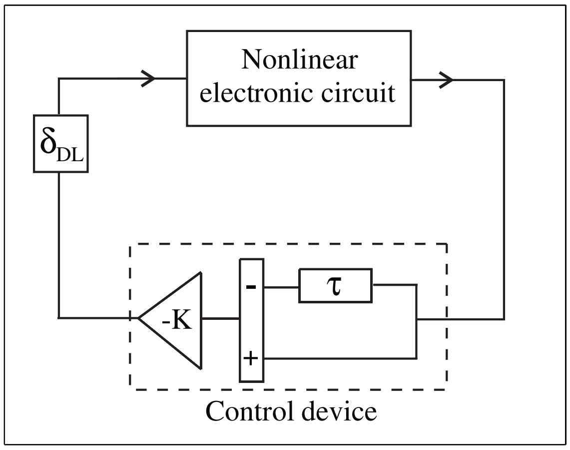

FIG. 3. Schematic view of the experimental set–up including nonlinear electronic circuit, control device, and additional delay line for adjustable control loop latency.

The plain nonlinear electronic circuit has been described in detail previously [10]. We adjusted its parameters in such a way that without control a Rössler type attractor appears which contains an unstable periodic orbit with period $\tau = 1.656\mu$s and Floquet frequency $\omega = \pi/\tau$. The control device consisted of a cascade of electronic delay lines with a limiting frequency of about 3MHz and several operational amplifiers acting as preamplifier, subtractor, or inverter. The device allowed to apply a control force of the form $F(t) = -K[U(t) - U(t-\tau)]$ with τ–range 10ns $\ldots$ 21μs. According to the period of the unstable orbit, the control delay was adjusted to $\tau = 1.656\mu$s. To investigate the effect of control loop latency we included a delay line between control device and feedback input. The control loop without additional delay line had a latency $\delta_0 = 37$ns. Therefore, by means of the additional delay line we could investigate latencies $\delta = \delta_0 + \delta_{DL}$, where δ_{DL} could be set in steps of 1ns.

To determine the region of successful control, i. e. the region where the control signal becomes as small as the electronic noise in the system, we swept the control amplitude K at fixed δ in order to obtain the control interval $[K_{min}(\delta), K_{max}(\delta)]$. The results for different δ values are shown in Fig.4. As can be clearly seen, the region of successful control strongly depends on the latency leading to the loss of control for $\delta/\tau \approx 11\%$. In order to compare these data with our analytical result (9) we choose the most simple form for the scaling factor $(-\tau\rho(\delta)) = A\sin(\pi\delta/\tau) + B\cos(\pi\delta/\tau)$ which is compatible with the antiperiodicity (8). A least square fit of the value of $K_{min}(\delta)$ leads to the values $A = -1.4$ and $B = 3.3$. In addition, the fit yields the Lyapunov exponent of the uncontrolled orbit $\lambda\tau = 1.6$. In view of our previous discussion one should, however, keep in mind that this value may differ from the exact one. According to expression (10) one obtains a critical value $\delta_c/\tau = 12.5\%$. Altogether, the quantitative coincidence of the experimental data with our analytical results is striking. Furthermore the shape of the control region agrees with previous experimental and numerical findings [12].

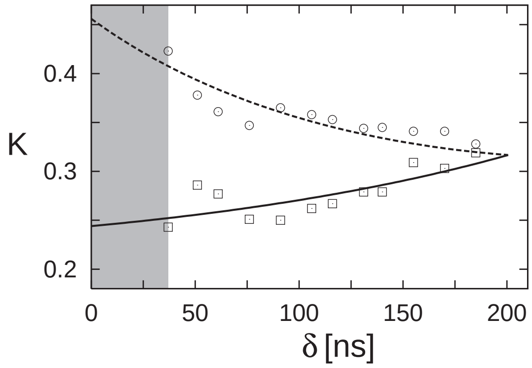

FIG. 4. Dependence of control interval on control loop latency, □: $K_{min}(\delta)$, ○: $K_{max}(\delta)$. The gray–shaded region is not accessible in our experiments due to the intrinsic latency δ_0. The lines are fits of the analytical result to the experimental data.

Conclusion – We have clarified the mechanism through which control loop latency affects the efficiency of delayed feedback control methods. The frequency splitting point which already limits the control interval for ordinary delayed feedback control is shifted in a way that the control interval shrinks in size. In our cases a latency of 10–20% of the period was sufficient to destroy the control at all. The obtained features can already be guessed from a superficial inspection of the eigenvalue equation (4), since the exponential originating from the latency reduces the effective control amplitude. Although our analysis was limited to flip orbits for simplicity, we think that the main features survive in the general case.

The analytical investigation of the eigenvalue equation was performed by employing a simple first–order Taylor series truncation. One may try to improve such an approximation by e. g. taking the large K asymptotic correctly into account. Nevertheless, as demonstrated by the experimental results, even the lowest order captures essential features of the influence of the control loop latency on the control interval. All the details of the system under consideration have been condensed to a single scalar $\rho(\delta)$, which rescales the control amplitude only. Finally we stress that, within our perturbative treatment, the maximal allowed latency has been related solely to properties of the unstable periodic orbit, namely the period and the Lyapunov exponent.

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