Weakly nonlinear analysis in spatially extended systems as a formal perturbation scheme

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(June 1, 1997)

The well known concept, to reduce the spatio–temporal dynamics beyond instabilities of trivial states to amplitude modulated patterns, is reviewed from the point of view of a formal perturbation expansion for general dissipative partial differential equations. For codimension one instabilities closed analytical formulas for all coefficients of the resulting amplitude equation are given, with no further restriction on the basic equations of motion. Both the autonomous and the explicitly time–dependent case are discussed. For the latter, the problem of strong resonances is addressed separately. The formal character of the expansion allows for an analysis of higher–codimension instabilities like the Turing–Hopf instability and for the discussion of principal limits of the amplitude approach in the present form.

I. INTRODUCTION

Pattern formation in dissipative systems under non–equilibrium conditions is a classical field of physical science and has in particular developed from hydrodynamic problems. In addition, this subject has become recently very popular in the context of optical, chemical, magnetic, and even biological systems (cf. ref. [1] and references theirin). The renewed interest was partially stimulated by developments in nonlinear dynamics.

Theoretical approaches rely strongly on analytical perturbation expansions, in order to reduce the equations of motion to dynamical relevant quantities. To some extent these concepts are limited to a neighbourhood of an instability of a simple state. Such concepts have proven to be powerful tools in the context of low–dimensional dynamical systems, even with quite mathematical rigour (e.g. [2]). For spatially extended systems with a large number of relevant degrees of freedom (i.e. the limit of large aspect ratio in the hydrodynamic context) such approaches have been introduced based on a multiple scale analysis [3,4]. They have been applied to a huge number of concrete examples and can even be found in textbooks (e.g. [5]). In fact the corresponding reduced description beyond simple instabilities, the Ginzburg–Landau equation, is very well known, although its properties are extremely complex and are a subject of intensive research.

Like in normal forms for differential equations, the details of the equations of motion, so to say the "physics", is to a large extent contained in the coefficients of the reduced equation. Hence it is desirable to have a closed expression at hand, to determine these quantities for an arbitrary equation of motion. Such formulas can be partially found in the literature [6–8], but unfortunately certain restrictions on the structure of the equation or the instability are imposed, which may limit the applicability of such expressions. It is one goal to fill this gap and to give these formulas for a large class of equations of motion including the case of explicitly time dependent systems. Hence for their evaluation, which only requires some simple algebraic computations, it is not necessary to perform the multiple scales analysis in each concrete case. In this sense our results apply easily to almost all evolution equations discussed in the physical context, considerably simplify the explicit computation of an amplitude equation, and show the common algebraic structure among the explicit expression for the coefficients as far as they are available in the literature.

In order to keep the presentation self–contained we review the complete derivation of amplitude equations within the well known concept of multiple scales analysis. Although our formulation follows the standard lines, we stress the following properties. The expansion can be understood as a formally exact procedure, and no physical assumptions (e.g. on scales), sometimes used to simplify calculations, are necessary. In addition, our approach respects the vector type structure inherent in the underlying equation of motion, and the final expressions can be understood as scalar

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quantities with respect to this structure. Hence with a suitable notation the full calculation is by no means more involved as for simple model equations. This observation has two important consequences. On the one hand it opens the possibility to analyse higher–codimension instabilities starting from physical equations of motion, by going to higher orders in the multiple scales analysis. On the other hand one can discuss the principal limits of the multiple scales approach in extended systems. In fact we will present one such example, which occurs within the class of explicitly time dependent systems. Finally our approach clearly distinguishes between the multiple scales expansion and separate approximations for the evaluation of the coefficients, which may be mixed if the concept is applied to a definite equation of motion.

For those readers who are not interested in technical details, section II summarises the notation and the results for the instability of a single mode, i.e. the explicit formulas for the coefficients of the Ginzburg–Landau equation. A brief discussion of the result is supplemented in section IV. The restriction to one spatial dimension is far from being only technical, since a formally satisfactory and sufficient general approach for spatially higher–dimensional rotationally symmetric systems is still missing, if one disregards the Newell–Whitehead–Segel treatment of nearly one–dimensional extended patterns. The technical derivation is reviewed in section III for autonomous systems. As an application to degenerated instabilities, the analysis for the Turing–Hopf instability together with the explicit formulas for the complete set of coupled amplitude equations is presented in section V. The peculiarities which arise in the presence of explicitly time–dependent equations are analysed in section VI. The phenomenon of strong resonances, which is associated with this situation, is discussed in section VII. Finally a few remarks on other higher–codimension instabilities and the generalisations for the discussion of spatially non–homogeneous situations are given.

II. BASIC NOTATION AND RESULT

We consider a physical system being invariant with respect to translations in space as well as in time. In order to deal with very general situations we allow for an N–component real field $\underline{\Phi}(x,t)$. A trivial, that means spatially homogeneous and time–independent, state should undergo an instability. Without loss of generality we may assume that its value is zero, so that the evolution equation reads

$$\frac{\partial \underline{\Phi}}{\partial t} = \mathcal{L}\underline{\Phi} + \mathcal{N}[\underline{\Phi}], \qquad \underline{\Phi} \in \mathbb{R}^N, \quad x \in \mathbb{R} \quad . \tag{1}$$

Here the linear operator $\mathcal{L}$ governs the instability and $\mathcal{N}$ denotes all the nonlinear contributions (cf. eqs.(10) and (12)).

We presuppose that on variation of the system parameters one mode with wavenumber q_c and frequency ω_c becomes unstable[1]. If we measure the deviation from this instability by the small quantity ε^2, then the linear operator can be cast into

$$\mathcal{L} = \mathcal{L}^{(0)} + \varepsilon^2 \mathcal{L}^{(2)} + \mathcal{O}(\varepsilon^4) \quad . \tag{2}$$

The most general expression for the linear operator at the threshold, which is compatible with the space–time translation invariance, reads

$$\mathcal{L}^{(0)}\underline{\psi} = \sum_\alpha \underline{\underline{L}}_\alpha \frac{\partial^\alpha}{\partial x^\alpha}\underline{\psi} \quad , \tag{3}$$

where the real matrices $\underline{\underline{L}}_\alpha$ take the vector character of the field into account[2]. A similar expression can be also written down for the second order contribution, but it is of no special use in the sequel. The (right–)eigenvalue problem of the operator (3) can be solved in terms of Fourier–modes

[1] Either the wavenumber or the frequency may vanish.

[2] The formal expression (3) for the linear operator was written down for convenience only, since a lot of evolution equations contain derivatives of finite order. Nevertheless the case of an infinite series, i.e. integral operators, is permitted too. It is evident from eqs.(4), (5) and the subsequent derivation, that our approach applies, if the kernel and the moments up to order three admit a Fourier transform.

$$
\begin{aligned}
\mathcal{L}^{(0)} e^{ikx} \underline{u}_k^{(\nu)} &= \lambda^{(\nu)}(k) e^{ikx} \underline{u}_k^{(\nu)} && (4)\\
\underline{\underline{L}}(k) \underline{u}_k^{(\nu)} &= \lambda^{(\nu)}(k) \underline{u}_k^{(\nu)} && (5)\\
\underline{\underline{L}}(k) := \sum_\alpha (ik)^\alpha \underline{\underline{L}}_\alpha & \quad . && (6)
\end{aligned}
$$

Here the index ν numbers the different branches of the eigenvalue problem. The eigenvalues as well as the vector part $\underline{u}_k^{(\nu)}$ of the eigenfunctions are completely determined by the complex N–dimensional algebraic equation (5). By presupposition all eigenvalues have a negative real part except for one branch $\nu = \nu_c$, where the real part vanishes at $k = q_c$[3]

$$
\mathrm{Re}\lambda^{(\nu_c)}(k) < 0 \ (|k| \neq |q_c|), \quad \lambda^{(\nu_c)}(q_c) =: i\omega_c, \quad \underline{u}_{q_c}^{(\nu_c)} =: \underline{u}_c \quad . \tag{7}
$$

Finally, we need for technical purposes the left–eigenvectors of the matrix (6)

$$
\underline{v}_k^{(\nu)*} \underline{\underline{L}}(k) = \underline{v}_k^{(\nu)*} \lambda^{(\nu)}(k) \quad , \tag{8}
$$

and especially the eigenvector at the threshold $\underline{v}_c := \underline{v}_{q_c}^{(\nu_c)}$.

Let us now turn to the nonlinear contributions. Of course they depend on the system parameters, that means on ε^2, too. But to the perturbation expansion only the expression at the threshold $\varepsilon = 0$ and terms of second or third order in the field amplitude contribute. Hence if we put

$$
\mathcal{N}[\underline{\psi}] = \mathcal{N}_2[\underline{\psi}] + \mathcal{N}_3[\underline{\psi}] + \mathcal{O}(\varepsilon^2, ||\underline{\psi}||^4) \quad , \tag{9}
$$

then the most general expressions of second and third order read

$$
\begin{aligned}
\mathcal{N}_2[\underline{\psi}] &= \sum_{\alpha\beta} C^{(\alpha\beta)} \left\{ \frac{\partial^\alpha \underline{\psi}}{\partial x^\alpha}, \frac{\partial^\beta \underline{\psi}}{\partial x^\beta} \right\} && (10)\\
\left(C^{(\alpha\beta)} \{\underline{u}, \underline{v}\} \right)_j &:= \sum_{lm} c_{jlm}^{(\alpha\beta)} u_l v_m && (11)\\
\mathcal{N}_3[\underline{\psi}] &= \sum_{\alpha\beta\gamma} D^{(\alpha\beta\gamma)} \left\{ \frac{\partial^\alpha \underline{\psi}}{\partial x^\alpha}, \frac{\partial^\beta \underline{\psi}}{\partial x^\beta}, \frac{\partial^\gamma \underline{\psi}}{\partial x^\gamma} \right\} && (12)\\
\left(D^{(\alpha\beta\gamma)} \{\underline{u}, \underline{v}, \underline{w}\} \right)_j &:= \sum_{lmn} d_{jlmn}^{(\alpha\beta\gamma)} u_l v_m w_n \quad . && (13)
\end{aligned}
$$

Here the real tensors (11) and (13) take the vector character of the equation into account[4].

In order to investigate the motion beyond the instability, we expand the solution of eq.(1) in terms of the small parameter ε, by taking explicitly the amplitude modulation of the marginally stable mode into account

$$
\underline{\Phi}(x,t) = \varepsilon \left[\underline{u}_c e^{iq_c x + i\omega_c t} A(\tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots) + \underline{u}_c^* e^{-iq_c x - i\omega_c t} A^*(\ldots) \right] + \varepsilon^2 \underline{\Phi}^{(2)} + \varepsilon^3 \underline{\Phi}^{(3)} + \mathcal{O}(\varepsilon^4) \quad . \tag{14}
$$

Here the abbreviations

$$
\tau_n := \varepsilon^n t, \quad \xi_n := \varepsilon^n x \tag{15}
$$

determine the scales of the slowly varying amplitude A. The evolution equation for A can be derived if we require that this representation does not contain secular contributions (cf. section III). This procedure results in the well known Ginzburg–Landau equation

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[3]For $q_c = 0$ a pair of complex conjugate eigenvalues occurs, because of symmetry. Then ν_c denotes one of these branches.

[4]In the sequel the symmetry relations

$$
C^{(\alpha\beta)}\{\underline{u}, \underline{v}\} = C^{(\beta\alpha)}\{\underline{v}, \underline{u}\} \quad D^{(\alpha\beta\gamma)}\{\underline{u}, \underline{v}, \underline{w}\} = D^{(\beta\alpha\gamma)}\{\underline{v}, \underline{u}, \underline{w}\} = \ldots
$$

are used.

$$\left(\frac{\partial}{\partial \tau_2} - v\frac{\partial}{\partial \xi_2}\right) A = \eta A + r|A|^2 A + D\frac{\partial A}{\partial \xi_1^2} \quad . \tag{16}$$

Here the convective velocity v and the diffusion coefficient D are given in terms of derivatives of the critical eigenvalue

$$v = \mathrm{Im}\frac{d\lambda^{(\nu_c)}(k)}{dk}\bigg|_{k=q_c} \tag{17}$$

$$D = -\frac{1}{2}\,\frac{d^2\lambda^{(\nu_c)}(k)}{dk^2}\bigg|_{k=q_c} \quad , \tag{18}$$

the coefficient of the cubic term is determined by the nonlinearities

$$r = \frac{(\underline{v}_c|\underline{\Gamma} + \underline{\Delta})}{(\underline{v}_c|\underline{u}_c)} \tag{19}$$

$$\underline{\Gamma} = 2\sum_{\alpha}(iq_c)^{\alpha} C^{(\alpha 0)}\left\{\underline{u}_c, 2\underline{\Gamma}_a\right\} + 2\sum_{\alpha\beta}(2iq_c)^{\beta}(-iq_c)^{\alpha} C^{(\alpha\beta)}\left\{\underline{u}_c^*, \underline{\Gamma}_b\right\} \tag{20}$$

$$\underline{\Gamma}_a := -\frac{1}{\underline{\underline{L}}(0)}\sum_{\alpha\beta}(iq_c)^{\alpha}(-iq_c)^{\beta} C^{(\alpha\beta)}\left\{\underline{u}_c, \underline{u}_c^*\right\} \tag{21}$$

$$\underline{\Gamma}_b := -\frac{1}{\underline{\underline{L}}(2q_c) - 2i\omega_c \underline{\underline{1}}}\sum_{\alpha\beta}(iq_c)^{\alpha}(iq_c)^{\beta} C^{(\alpha\beta)}\left\{\underline{u}_c, \underline{u}_c\right\} \tag{22}$$

$$\underline{\Delta} = 3\sum_{\alpha\beta\gamma}(iq_c)^{\alpha}(iq_c)^{\beta}(-iq_c)^{\gamma} D^{(\alpha\beta\gamma)}\left\{\underline{u}_c, \underline{u}_c, \underline{u}_c^*\right\} \quad , \tag{23}$$

and the coefficient of the linear term is given by the matrix element of the perturbation $\mathcal{L}^{(2)}$

$$\eta = \frac{q_c/(2\pi)\int_0^{2\pi/q_c}\left(\underline{v}_c e^{iq_c x}|\mathcal{L}^{(2)}|\underline{u}_c e^{iq_c x}\right)\,dx}{(\underline{v}_c|\underline{u}_c)} \quad . \tag{24}$$

The integral[5] just picks out the Fourier component with wave number q_c. $(.|.)$ denotes the usual scalar product in $\mathbb{C}^N$. For convenience appendix C contains the evaluation of these expressions for the Maxwell–Bloch equations as an example.

III. DERIVATION OF THE AMPLITUDE EQUATION

One inserts the expression (14) into eq.(1) and expands order by order taking the notation (2), (3), (9), (10), and (12) into account.

To first order in ε only the linear operator contributes. Since the amplitude A acts as a constant at this order one obtains

$$i\omega_c \underline{u}_c e^{iq_c x + i\omega_c t} A - i\omega_c \underline{u}_c^* e^{-iq_c x - i\omega_c t} A^* = \underline{\underline{L}}(q_c)\underline{u}_c e^{iq_c x + i\omega_c t} A + \underline{\underline{L}}(-q_c)\underline{u}_c^* e^{-iq_c x - i\omega_c t} A^* \quad . \tag{25}$$

By virtue of the relation $\underline{\underline{L}}(k)^* = \underline{\underline{L}}(-k)$ this expression is nothing else but the eigenvalue equation (5) of the critical mode (cf. (7)).

To second order in ε one gets

$$\begin{aligned} &\frac{\partial \underline{\Phi}^{(2)}}{\partial t} + \underline{u}_c e^{iq_c x + i\omega_c t}\frac{\partial A}{\partial \tau_1} + \underline{u}_c^* e^{-iq_c x - i\omega_c t}\frac{\partial A^*}{\partial \tau_1} \\ &= \left(\mathcal{L}^{(0)}\underline{\Phi}^{(2)}\right)^{[0]} + \left(\mathcal{L}^{(0)}\underline{u}_c e^{iq_c x + i\omega_c t} A\right)^{[1]} + \left(\mathcal{L}^{(0)}\underline{u}_c^* e^{-iq_c x - i\omega_c t} A^*\right)^{[1]} + \left(\mathcal{N}_2[\underline{\Phi}]\right)^{[2]} \quad . \end{aligned} \tag{26}$$

[5]The case $q_c = 0$, in which only the scalar products appear, is captured by this notations as the formal limit $q_c \to 0$.

Here the additional superscript characters indicate at which order in ε the expression in the bracket has to be evaluated.

First let us consider the linear terms. From the identity

$$\left(\frac{\partial^\alpha}{\partial x^\alpha} e^{iq_c x} A\right)^{[1]} = \alpha (iq_c)^{\alpha-1} e^{iq_c x} \frac{\partial A}{\partial \xi_1} \tag{27}$$

and the definition (3), the relation

$$\left(\mathcal{L}^{(0)} \underline{u}_c e^{iq_c x + i\omega_c t} A\right)^{[1]} = -i\underline{\underline{L}}'(q_c) \underline{u}_c e^{iq_c x + i\omega_c t} \frac{\partial A}{\partial \xi_1} \tag{28}$$

follows. In general $\underline{u}_k^{(\nu_c)}$ depends on the wavenumber, so that $\underline{u}_c$ is not an eigenvector of the derivative $\underline{\underline{L}}'(q_c)$. However, as shown in appendix A, we can consider this special case without imposing any restriction on the validity of our results. Although this step is by no means essential, it helps to simplify considerably the following computations. By taking the derivative of the eigenvalue equation (5) with respect to k at $k = q_c$ and recalling that the real part has a maximum at q_c, one obtains

$$\underline{\underline{L}}'(q_c) \underline{u}_c = iv \underline{u}_c \tag{29}$$

taking the definition (17) into account. Hence eq.(26) simplifies to

$$\frac{\partial \underline{\Phi}^{(2)}}{\partial t} = \left(\mathcal{L}^{(0)} \underline{\Phi}^{(2)}\right)^{[0]} + \left[\underline{u}_c e^{iq_c x + i\omega_c t} \left(v \frac{\partial A}{\partial \xi_1} - \frac{\partial A}{\partial \tau_1}\right) + \underline{u}_c^* e^{-iq_c x - i\omega_c t} \left(v \frac{\partial A^*}{\partial \xi_1} - \frac{\partial A^*}{\partial \tau_1}\right)\right] + (\mathcal{N}_2[\underline{\Phi}])^{[2]} \quad . \tag{30}$$

The nonlinear term is easily evaluated with the help of eqs.(10) and (14)

$$(\mathcal{N}_2[\underline{\Phi}])^{[2]} = -2\underline{\underline{L}}(0)\underline{\Gamma}_a |A|^2 - [\underline{\underline{L}}(2q_c) - 2i\omega_c \underline{\underline{1}}]\underline{\Gamma}_b e^{2iq_c x + 2i\omega_c t} A^2 - [\underline{\underline{L}}(-2q_c) + 2i\omega_c \underline{\underline{1}}]\underline{\Gamma}_b^* e^{-2iq_c x - 2i\omega_c t} (A^*)^2 \quad , \tag{31}$$

if the abbreviations (21) and (22) are used. We have to check the secular condition (B4), to determine the solution $\underline{\Phi}^{(2)}$. Only the modes $\pm e^{iq_c x + i\omega_c t}$ contribute, because either the wavenumber or the frequency do not vanish. The evaluation of this condition yields

$$0 = \left(\frac{\partial}{\partial \tau_1} - v \frac{\partial}{\partial \xi_1}\right) A \quad . \tag{32}$$

As a consequence the second and third term on the right hand side of eq.(30) vanish. For the solution $\underline{\Phi}^{(2)}$ one obtains discarding transients (cf. eq.(B2))

$$\underline{\Phi}^{(2)} = 2\underline{\Gamma}_a |A|^2 + \underline{\Gamma}_b e^{2iq_c x + 2i\omega_c t} A^2 + \underline{\Gamma}_b^* e^{-2iq_c x - 2i\omega_c t} (A^*)^2 + \underline{u}_c e^{iq_c x + i\omega_c t} B(\tau_1, \ldots, \xi_1, \ldots) + \underline{u}_c^* e^{-iq_c x - i\omega_c t} B^*(\ldots) \quad , \tag{33}$$

where the constant of integration may of course depend on the slower scales.

To third order in ε the equation of motion (1) reads

$$\begin{aligned} &\frac{\partial \underline{\Phi}^{(3)}}{\partial t} + \left(\frac{\partial \underline{\Phi}^{(2)}}{\partial t}\right)^{[1]} + \underline{u}_c e^{iq_c x + i\omega_c t} \frac{\partial A}{\partial \tau_2} + \underline{u}_c^* e^{-iq_c x - i\omega_c t} \frac{\partial A^*}{\partial \tau_2} \\ &= \mathcal{L}^{(0)} \underline{\Phi}^{(3)} + \left(\mathcal{L}^{(0)} \underline{\Phi}^{(2)}\right)^{[1]} + \left(\mathcal{L}^{(0)} \underline{u}_c e^{iq_c x + i\omega_c t} A\right)^{[2]} + \left(\mathcal{L}^{(0)} \underline{u}_c^* e^{-iq_c x - i\omega_c t} A^*\right)^{[2]} \\ &+ \left(\mathcal{L}^{(2)} \underline{u}_c e^{iq_c x + i\omega_c t}\right) \cdot A + \left(\mathcal{L}^{(2)} \underline{u}_c^* e^{-iq_c x - i\omega_c t}\right) \cdot A^* + (\mathcal{N}_2[\underline{\Phi}])^{[3]} + (\mathcal{N}_3[\underline{\Phi}])^{[3]} \quad . \end{aligned} \tag{34}$$

From eq.(33) we have

$$\left(\frac{\partial \underline{\Phi}^{(2)}}{\partial t}\right)^{[1]} = \underline{u}_c e^{iq_c x + i\omega_c t} \frac{\partial B}{\partial \tau_1} + \underline{u}_c^* e^{-iq_c x - i\omega_c t} \frac{\partial B^*}{\partial \tau_1} + \cdots \quad . \tag{35}$$

Here and in the remaining part of this section $\cdots$ indicate Fourier–modes (with wave vector $\pm 2q_c, \pm 3q_c$ or frequency $\pm 2\omega_c, \pm 3\omega_c$), which will not contribute to the secular condition. In the same way we obtain using eq.(28)

$$\left(\mathcal{L}^{(0)}\underline{\Phi}^{(2)}\right)^{[1]} = -i\underline{\underline{L}}'(q_c)\underline{u}_c e^{iq_cx+i\omega_ct}\frac{\partial B}{\partial \xi_1} - i\underline{\underline{L}}'(-q_c)\underline{u}_c^* e^{-iq_cx-i\omega_ct}\frac{\partial B^*}{\partial \xi_1} + \cdots \tag{36}$$

With the identity analogous to eq.(27)

$$\left(\frac{\partial^\alpha}{\partial x^\alpha}e^{iq_cx+i\omega_ct}A\right)^{[2]} = \alpha(iq_c)^{\alpha-1}e^{iq_cx+i\omega_ct}\frac{\partial A}{\partial \xi_2} + \frac{\alpha(\alpha-1)}{2}(iq_c)^{\alpha-2}e^{iq_cx+i\omega_ct}\frac{\partial^2 A}{\partial \xi_1^2} \tag{37}$$

one has

$$\left(\mathcal{L}^{(0)}\underline{u}_c e^{iq_cx+i\omega_ct}A\right)^{[2]} = -i\underline{\underline{L}}'(q_c)\underline{u}_c e^{iq_cx+i\omega_ct}\frac{\partial A}{\partial \xi_2} + \frac{(-i)^2}{2}\underline{\underline{L}}''(q_c)\underline{u}_c e^{iq_cx+i\omega_ct}\frac{\partial^2 A}{\partial \xi_1^2} \quad . \tag{38}$$

For the evaluation of the nonlinear contributions we again keep in mind that only the resonant Fourier–modes have to be considered. Then

$$\begin{aligned}(\mathcal{N}_2[\underline{\Phi}])^{[3]} &= 2\sum_{\alpha\beta} C^{(\alpha\beta)}\left\{(iq_c)^\alpha \underline{u}_c e^{iq_cx+i\omega_ct}A + (-iq_c)^\alpha \underline{u}_c^* e^{-iq_cx-i\omega_ct}A^*, \left(\frac{\partial^\beta \underline{\Phi}^{(2)}}{\partial x^\beta}\right)^{[0]}\right\} + \cdots \\ &= \underline{\Gamma}e^{iq_cx+i\omega_ct}|A|^2A + \underline{\Gamma}^* e^{-iq_cx-i\omega_ct}|A|^2A^* + \cdots \quad ,\end{aligned} \tag{39}$$

where for the evaluation eq.(33) and the abbreviation (20) have been used. In the same way one obtains

$$(\mathcal{N}_3[\underline{\Phi}])^{[3]} = \underline{\Delta}e^{iq_cx+i\omega_ct}|A|^2A + \underline{\Delta}^* e^{-iq_cx-i\omega_ct}|A|^2A^* + \cdots \tag{40}$$

using the abbreviation (23).

If we now collect eqs.(35), (36), and (38) the evolution equation (34) reads

$$\begin{aligned}\frac{\partial \underline{\Phi}^{(3)}}{\partial t} &= \left(\mathcal{L}^{(0)}\underline{\Phi}^{(3)}\right)^{[0]} + \underline{u}_c e^{iq_cx+i\omega_ct}\left(v\frac{\partial B}{\partial \xi_1} - \frac{\partial B}{\partial \tau_1}\right) + \underline{u}_c^* e^{-iq_cx-i\omega_ct}\left(v\frac{\partial B^*}{\partial \xi_1} - \frac{\partial B^*}{\partial \tau_1}\right) \\ &+ \underline{u}_c e^{iq_cx+i\omega_ct}\left(v\frac{\partial A}{\partial \xi_2} - \frac{\partial A}{\partial \tau_2}\right) + \underline{u}_c^* e^{-iq_cx-i\omega_ct}\left(v\frac{\partial A^*}{\partial \xi_2} - \frac{\partial A^*}{\partial \tau_2}\right) \\ &- \frac{1}{2}\left.\frac{d^2\lambda^{(\nu_c)}(k)}{dk^2}\right|_{k=q_c}\underline{u}_c e^{iq_cx+i\omega_ct}\frac{\partial^2 A}{\partial \xi_1^2} - \frac{1}{2}\left.\frac{d^2\lambda^{(\nu_c)*}(k)}{dk^2}\right|_{k=q_c}\underline{u}_c^* e^{-iq_cx-i\omega_ct}\frac{\partial^2 A^*}{\partial \xi_1^2} \\ &+ \left(\mathcal{L}^{(2)}\underline{u}_c e^{iq_cx+i\omega_ct}\right)\cdot A + \left(\mathcal{L}^{(2)}\underline{u}_c^* e^{-iq_cx-i\omega_ct}\right)\cdot A^* \\ &+ (\mathcal{N}_2[\underline{\Phi}])^{[3]} + (\mathcal{N}_3[\underline{\Phi}])^{[3]} + \cdots \quad ,\end{aligned} \tag{41}$$

where the nonlinear terms are given by eqs.(39) and (40). In addition we have used eq.(29) and the analogous relation for the second derivative.

Only the terms written explicitly will contribute to the secular condition, for the reasons mentioned above. Eq.(B4) yields

$$0 = \left(v\frac{\partial B}{\partial \xi_1} - \frac{\partial B}{\partial \tau_1}\right) + \left(v\frac{\partial A}{\partial \xi_2} - \frac{\partial A}{\partial \tau_2}\right) + D\frac{\partial^2 A}{\partial \xi_1^2} + \eta A + r|A|^2A \tag{42}$$

if one takes the abbreviations (18), (19), and (24) into account and recalls, that $\mathcal{L}^{(2)}$ is invariant with respect to translations in space and time.

Finally one has to separate the amplitudes A and B from each other. For that purpose apply $(\partial/\partial\tau_1 - v\partial/\partial\xi_1)$ to eq.(42). Then

$$0 = \left(\frac{\partial}{\partial \tau_1} - v\frac{\partial}{\partial \xi_1}\right)^2 B \tag{43}$$

holds, if we use secular condition (32) of the preceding order. Since the general solution of this equation is given by $B = f_0(\xi_1 + v\tau_1) + (\xi_1 - v\tau_1)f_1(\xi_1 + v\tau_1)$, but B must not contain a secular contribution, one has $f_1 \equiv 0$. Hence

$$0 = \left(\frac{\partial}{\partial \tau_1} - v\frac{\partial}{\partial \xi_1}\right) B \quad , \tag{44}$$

and eq.(42) results in eq.(16).

IV. DISCUSSION

The analysis of the preceding section has shown, that the dynamics beyond an instability of a single mode is generically described by a complex Ginzburg–Landau equation, if the underlying dynamics is autonomous. Since the solutions of this reduced equation are bounded for $\mathrm{Re}D > 0$ and $\mathrm{Re}\, r < 0$ (cf. [9,10]), the motion is correctly described at least on time scales $t \sim \varepsilon^{-2}$. It is not the aim of this article to go into the details of the properties of the amplitude equation, which are itself a field of current research. Let us only mention that the change in sign of $\mathrm{Re}D$ or $\mathrm{Re}\, r$ are higher–codimension instabilities, i.e. a kind of Eckhaus instability and the transition from super– to sub–critical behaviour respectively.

Taking symmetries of the underlying equations of motion into account, the apparently complicated expressions for the coefficients of the amplitude equation may be simplified considerably. A frequent constraint in this direction is a system, being symmetric with respect to space–inversion. Let us restrict therefore the following discussion to this case. As the main consequence of the symmetry the matrix (6) determining the stability is always real. Hence one real or two complex conjugated critical eigenvalues for the unique critical mode occurs.

Consider first the case of a non–vanishing critical wave number $q_c \neq 0$, which is sometimes called a soft–mode instability. Then the restriction to one critical mode implies a vanishing frequency, even for wave-numbers in a neighbourhood of q_c. Hence the derivatives of the spectrum and the eigenvectors are real quantities. But then the convective velocity vanishes and the remaining coefficients of the amplitude equation (cf. eqs.(18)–(23)) are real, since only derivatives of even order occur.

If the critical wave vector vanishes, the frequency must be finite $\omega_c \neq 0$. Such a situation is sometimes called a hard–mode instability. Since the eigenvectors are now complex, the coefficients do not reduce to real numbers even in the presence of inversion symmetry. But the convective velocity (17) vanishes for that reason. In addition, since $q_c = 0$, only the spatially homogeneous contributions, i.e. one summand, enter the formulas for the cubic coefficient (cf. eqs.(19)–(23)). Of course r reduces to the expression known from the simple Hopf bifurcation (cf. [2, p.152f]).

V. TURING–HOPF INSTABILITY

As stated in the previous section soft– and hard–mode instabilities are typical, i.e. of codimension one, in systems which are symmetric with respect to space–inversion. The corresponding codimension two situation, where both instabilities occur simultaneously has recently attracted considerable interest even from the experimental point of view [11]. For that reason we think it is useful to present the general analytical expressions for the coefficients of the corresponding coupled amplitude equations, which govern the dynamics beyond the instability.

Again we start our analysis from the general autonomous evolution equation (1) but impose in addition the inversion symmetry. Since a soft– and a hard–mode become unstable simultaneously we presuppose that instead of eq.(7) the spectrum obeys $\lambda^{(\nu_c)}(q_c) = 0$ and $\lambda^{(\nu_c)}(0) = i\omega_c$. For the weakly nonlinear analysis one has to take both modes into account, where the amplitudes may depend on the slower spatial and temporal scales

$$\underline{\Phi}(x,t) = \varepsilon\left[\underline{u}_c^S e^{iq_cx} A^S + \underline{u}_c^S e^{-iq_cx} A^{S\,*} + \underline{u}_c^H e^{i\omega_ct} A^H + \underline{u}_c^{H\,*} e^{-i\omega_ct} A^{H\,*}\right] + \varepsilon^2 \underline{\Phi}^{(2)} + \varepsilon^3 \underline{\Phi}^{(3)} + \mathcal{O}(\varepsilon^4) \quad . \tag{45}$$

Since $\underline{\underline{L}}(q_c)$ is real the vector part $\underline{u}_c^S := \underline{u}_{q_c}^{(\nu_c)}$ of the soft mode is real too. In analogy to section II the first order leads to the instability condition. For the second order we obtain

$$\frac{\partial \underline{\Phi}^{(2)}}{\partial t} = \left(\mathcal{L}^{(0)} \underline{\Phi}^{(2)}\right)^{[0]} - \underline{u}_c^S e^{iq_cx} \frac{\partial A^S}{\partial \tau_1} - \underline{u}_c^S e^{-iq_cx} \frac{\partial A^{S\,*}}{\partial \tau_1} - \underline{u}_c^H e^{i\omega_ct} \frac{\partial A^H}{\partial \tau_1} - \underline{u}_c^{H\,*} e^{-i\omega_ct} \frac{\partial A^{H\,*}}{\partial \tau_1} + (\mathcal{N}_2[\underline{\Phi}])^{[2]} \tag{46}$$

if we again make use of the fact, that the eigenvectors can be assumed to be independent of the wavenumber, and that the first derivatives of the eigenvalues vanish by inversion symmetry. That part of eq.(46) which is linear in the amplitudes is nothing else but the identical copy of eq.(30) for the two modes. The nonlinear part reads

$$\begin{aligned}
(\mathcal{N}_2[\underline{\Phi}])^{[2]} = & -2\underline{\underline{L}}(0)\underline{\Gamma}_a^S |A^S|^2 - \underline{\underline{L}}(2q_c)\underline{\Gamma}_b^S e^{2iq_cx} (A^S)^2 - \underline{\underline{L}}(-2q_c)\underline{\Gamma}_b^{S\,*} e^{-2iq_cx} (A^{S\,*})^2 \\
& - 2\underline{\underline{L}}(0)\underline{\Gamma}_a^H |A^H|^2 - [\underline{\underline{L}}(0) - 2i\omega_c\underline{\underline{1}}]\underline{\Gamma}_b^H e^{2i\omega_ct} (A^H)^2 - [\underline{\underline{L}}(0) + 2i\omega_c\underline{\underline{1}}]\underline{\Gamma}_b^{H\,*} e^{-2i\omega_ct} (A^{H\,*})^2 \\
& - 2[\underline{\underline{L}}(q_c) - i\omega_c\underline{\underline{1}}]\underline{\Gamma}_c e^{iq_cx + i\omega_ct} A^H A^S - 2[\underline{\underline{L}}(q_c) + i\omega_c\underline{\underline{1}}]\underline{\Gamma}_c^* e^{-iq_cx - i\omega_ct} A^{H\,*} A^{S\,*} \\
& - 2[\underline{\underline{L}}(q_c) - i\omega_c\underline{\underline{1}}]\underline{\Gamma}_c e^{-iq_cx + i\omega_ct} A^H A^{S\,*} - 2[\underline{\underline{L}}(q_c) + i\omega_c\underline{\underline{1}}]\underline{\Gamma}_c^* e^{iq_cx - i\omega_ct} A^{H\,*} A^S \quad .
\end{aligned} \tag{47}$$

The first two lines coincide with the corresponding expression from the codimension one case (cf. eq.(31)). The coefficients $\underline{\Gamma}_{a/b}^S$ and $\underline{\Gamma}_{a/b}^H$ are given by eqs.(21) and (22) evaluated for the soft– and hard–mode respectively. The remaining terms describe the interaction of the modes with the coefficient being given by

$$\underline{\Gamma}_c := -\frac{1}{\underline{\underline{L}}(q_c) - i\omega_c \underline{\underline{1}}} \sum_\beta (iq_c)^\beta C^{(0\beta)} \{\underline{u}_c^H, \underline{u}_c^S\} \quad . \tag{48}$$

Since two modes are critical, the secular condition (B4) has to be evaluated twice and yields

$$0 = \frac{\partial A^S}{\partial \tau_1}, \qquad 0 = \frac{\partial A^H}{\partial \tau_1} \quad . \tag{49}$$

The the solution of second order, discarding transients, reads

$$\begin{aligned}
\underline{\Phi}^{(2)} &= 2\underline{\Gamma}_a^S |A^S|^2 + \underline{\Gamma}_b^S e^{2iq_c x}(A^S)^2 + \underline{\Gamma}_b^{S\,*} e^{-2iq_c x}(A^{S\,*})^2 + \underline{u}_c^S e^{iq_c x} B^S + \underline{u}_c^{S\,*} e^{-iq_c x} B^{S\,*} \\
&+ 2\underline{\Gamma}_a^H |A^H|^2 + \underline{\Gamma}_b^H e^{2i\omega_c t}(A^H)^2 + \underline{\Gamma}_b^{H\,*} e^{-2i\omega_c t}(A^{H\,*})^2 + \underline{u}_c^H e^{i\omega_c t} B^H + \underline{u}_c^{H\,*} e^{-i\omega_c t} B^{H\,*} \\
&+ 2\underline{\Gamma}_c e^{iq_c x + i\omega_c t} A^H A^S + 2\underline{\Gamma}_c^* e^{-iq_c x - i\omega_c t} A^{H\,*} A^{S\,*} + 2\underline{\Gamma}_c e^{-iq_c x + i\omega_c t} A^H A^{S\,*} + 2\underline{\Gamma}_c^* e^{iq_c x - i\omega_c t} A^{H\,*} A^S \quad .
\end{aligned} \tag{50}$$

where the constants of integration B^S and B^H depend on the slower scales. At third order we now obtain

$$\begin{aligned}
\frac{\partial \underline{\Phi}^{(3)}}{\partial t} &= \left(\mathcal{L}^{(0)} \underline{\Phi}^{(3)}\right)^{[0]} - \underline{u}_c^S e^{iq_c x} \frac{\partial B^S}{\partial \tau_1} - \underline{u}_c^S e^{-iq_c x} \frac{\partial B^{S\,*}}{\partial \tau_1} - \underline{u}_c^S e^{iq_c x} \frac{\partial A^S}{\partial \tau_2} - \underline{u}_c^S e^{-iq_c x} \frac{\partial A^{S\,*}}{\partial \tau_2} \\
&- \frac{1}{2} \left.\frac{d^2 \lambda^{(\nu_c)}(k)}{dk^2}\right|_{k=q_c} \underline{u}_c^S e^{iq_c x} \frac{\partial^2 A^S}{\partial \xi_1^2} - \frac{1}{2} \left.\frac{d^2 \lambda^{(\nu_c)}(k)}{dk^2}\right|_{k=q_c} \underline{u}_c^S e^{-iq_c x} \frac{\partial^2 A^{S\,*}}{\partial \xi_1^2} \\
&+ \left(\mathcal{L}^{(2)} \underline{u}_c^S e^{iq_c x}\right) \cdot A^S + \left(\mathcal{L}^{(2)} \underline{u}_c^S e^{-iq_c x}\right) \cdot A^{S\,*} \\
&- \underline{u}_c^H e^{i\omega_c t} \frac{\partial B^H}{\partial \tau_1} - \underline{u}_c^{H\,*} e^{-i\omega_c t} \frac{\partial B^{H\,*}}{\partial \tau_1} - \underline{u}_c^H e^{i\omega_c t} \frac{\partial A^H}{\partial \tau_2} - \underline{u}_c^{H\,*} e^{-i\omega_c t} \frac{\partial A^{H\,*}}{\partial \tau_2} \\
&- \frac{1}{2} \left.\frac{d^2 \lambda^{(\nu_c)}(k)}{dk^2}\right|_{k=0} \underline{u}_c^H e^{i\omega_c t} \frac{\partial^2 A^H}{\partial \xi_1^2} - \frac{1}{2} \left.\frac{d^2 \lambda^{(\nu_c)*}(k)}{dk^2}\right|_{k=0} \underline{u}_c^{H\,*} e^{-i\omega_c t} \frac{\partial^2 A^{H\,*}}{\partial \xi_1^2} \\
&+ \left(\mathcal{L}^{(2)} \underline{u}_c^H e^{i\omega_c t}\right) \cdot A^H + \left(\mathcal{L}^{(2)} \underline{u}_c^{H\,*} e^{-i\omega_c t}\right) \cdot A^{H\,*} \\
&+ (\mathcal{N}_2[\underline{\Phi}])^{[3]} + (\mathcal{N}_3[\underline{\Phi}])^{[3]} + \cdots \quad .
\end{aligned} \tag{51}$$

This apparently complicated expression contains the linear part of the two modes which is already known from the codimension one analysis (cf. eq.(41)). The nonlinear contributions read

$$\begin{aligned}
(\mathcal{N}_2[\underline{\Phi}])^{[3]} &= \underline{\Gamma}^S e^{iq_c x} |A^S|^2 A^S + \underline{\Gamma}^{S\,*} e^{-iq_c x} |A^S|^2 A^{S\,*} + \underline{\Gamma}^H e^{i\omega_c t} |A^H|^2 A^H + \underline{\Gamma}^{H\,*} e^{-i\omega_c t} |A^H|^2 A^{H\,*} \\
&+ \underline{\gamma}^S e^{iq_c x} |A^H|^2 A^S + \underline{\gamma}^{S\,*} e^{-iq_c x} |A^H|^2 A^{S\,*} + \underline{\gamma}^H e^{i\omega t} |A^S|^2 A^H + \underline{\gamma}^{H\,*} e^{-i\omega t} |A^S|^2 A^{H\,*} + \cdots
\end{aligned} \tag{52}$$

if we restrict the presentation to terms which will contribute to the secular condition. Again the first line describes the contribution already known from the analysis presented above, with the coefficients $\underline{\Gamma}^S$ and $\underline{\Gamma}^H$ being determined by eqs.(20)–(22) evaluated for the soft– and hard–mode. The new coefficients which mediate the interaction are given by

$$\underline{\gamma}^S := 2\sum_\beta (iq_c)^\beta C^{(0\beta)} \{\underline{u}_c^{H\,*}, 2\underline{\Gamma}_c\} + 2\sum_\beta (iq_c)^\beta C^{(0\beta)} \{\underline{u}_c^H, 2\underline{\Gamma}_c^*\} + 2\sum_\alpha (iq_c)^\alpha C^{(\alpha 0)} \{\underline{u}_c^S, 2\underline{\Gamma}_a^H\} \qquad (\in \mathbb{R}^N) \tag{53}$$

$$\underline{\gamma}^H := 2\sum_{\alpha\beta} (-iq_c)^\alpha (iq_c)^\beta C^{(\alpha\beta)} \{\underline{u}_c^S, 2\underline{\Gamma}_c\} + 2\sum_{\alpha\beta} (iq_c)^\alpha (-iq_c)^\beta C^{(\alpha\beta)} \{\underline{u}_c^S, 2\underline{\Gamma}_c\} + 2C^{(00)} \{\underline{u}_c^H, 2\underline{\Gamma}_a^S\} \quad . \tag{54}$$

In the same way the other term reads

$$\begin{aligned}
(\mathcal{N}_3[\underline{\Phi}])^{[3]} &= \underline{\Delta}^S e^{iq_c x} |A^S|^2 A^S + \underline{\Delta}^{S\,*} e^{-iq_c x} |A^S|^2 A^{S\,*} + \underline{\Delta}^H e^{i\omega_c t} |A^H|^2 A^H + \underline{\Delta}^{H\,*} e^{-i\omega_c t} |A^H|^2 A^{H\,*} \\
&+ \underline{\delta}^S e^{iq_c x} |A^H|^2 A^S + \underline{\delta}^{S\,*} e^{-iq_c x} |A^H|^2 A^{S\,*} + \underline{\delta}^H e^{i\omega_c t} |A^S|^2 A^H + \underline{\delta}^{H\,*} e^{-i\omega_c t} |A^S|^2 A^{H\,*} + \cdots \quad ,
\end{aligned} \tag{55}$$

where the interaction coefficients are

$$\underline{\delta}^S := 6\sum_\gamma (iq_c)^\gamma D^{(00\gamma)} \{\underline{u}_c^H, \underline{u}_c^{H\,*}, \underline{u}_c^S\} \qquad (\in \mathbb{R}^N) \tag{56}$$

$$\underline{\delta}^H := 6\sum_{\beta\gamma} (-iq_c)^\beta (iq_c)^\gamma D^{(0\beta\gamma)} \{\underline{u}_c^H, \underline{u}_c^S, \underline{u}_c^S\} \quad . \tag{57}$$

If one evaluates the secular conditions arising from eqs.(51), (52), and (55) and separates the amplitudes A and B one finally obtains the coupled set of amplitude equations

$$\frac{\partial A^S}{\partial \tau_2} = \eta^S A^S + r^S |A^S|^2 A^S + s^S |A^H|^2 A^S + D^S \frac{\partial A^S}{\partial \xi_1^2} \tag{58}$$

$$\frac{\partial A^H}{\partial \tau_2} = \eta^H A^H + r^H |A^H|^2 A^H + s^H |A^S|^2 A^H + D^H \frac{\partial A^H}{\partial \xi_1^2} \quad , \tag{59}$$

with the coupling coefficients being given by

$$s^S = \frac{(\underline{v}_c^S | \underline{\gamma}^S + \underline{\delta}^S)}{(\underline{v}_c^S | \underline{u}_c^S)} \quad (\in \mathbb{R}), \qquad s^H = \frac{(\underline{v}_c^H | \underline{\gamma}^H + \underline{\delta}^H)}{(\underline{v}_c^H | \underline{u}_c^H)} \tag{60}$$

and eqs.(53), (54), (56), (57). The remaining coefficients are of course not changed compared to the codimension one case and are given by eqs.(18)–(24) evaluated for the soft– and the hard–mode.

We have obtained the well known set of coupled real and complex Ginzburg–Landau equations [12], but with the general and closed analytical formulas for its coefficients. The simplicity of the whole derivation for this quite complicated instability again justifies the slight amount of formalism presented in the preceding sections.

VI. EXPLICITLY TIME–DEPENDENT EQUATIONS

Let us now return to the case of a single unstable mode and consider the case that the equations of motion (1), that means the linear operator $\mathcal{L}$ and the nonlinear contributions $\mathcal{N}$, possess an additional explicit time–dependence with period T and frequency $\Omega = 2\pi/T$ respectively. Such a dependence may arise in the presence of external time–dependent fields. But one should also keep in mind, that a time–dependency may be introduced also by the trivial state, if the equation of motion is cast into the form (1). For the derivation of an amplitude equation the steps of the preceding section can be applied. The calculation is quite similar and we use the same symbols, even if they are now T–periodic functions of time. However, two main formal differences occur. On the one hand the stability of the trivial state is governed by a Floquet– instead of an eigenvalue problem, that means that the eigenvectors itself are T–periodic in time and obey

$$\mu^{(\nu)}(k) \underline{u}_k^{(\nu)}(t) + \dot{\underline{u}}_k^{(\nu)}(t) = \underline{\underline{L}}(k,t) \underline{u}_k^{(\nu)}(t) \tag{61}$$

$$\underline{v}_k^{(\nu)*}(t) \mu^{(\nu)}(k) - \dot{\underline{v}}_k^{(\nu)*}(t) = \underline{v}_k^{(\nu)*}(t) \underline{\underline{L}}(k,t) \quad . \tag{62}$$

Here $\underline{\underline{L}}(k,t)$ is defined by eq.(6) with $\underline{\underline{L}}_\alpha$ being explicitly time–dependent, and the imaginary part of the Floquet–exponents is restricted to the first Brillouin zone, $\mathrm{Im}\mu^{(\nu)}(k) \in [-\Omega/2, \Omega/2]$. On the other hand the matrix coefficients in the abbreviations (21) and (22), which are actually some propagators of the linear equation, are replaced by the corresponding quantity of the time–dependent equation. We will present the details below.

The main physical difference comes from the fact, that one has to pay attention to strong resonances, which are known to be crucial already in the corresponding spatially homogeneous situation [13]. For that reason it is worthwhile to consider the explicitly time–dependent case separately.

We first consider the situation without strong resonances. Let us assume that either the critical wave number q_c does not vanish or that the frequency ω_c does not obey a strong resonance condition, i.e.

$$\omega_c \neq \pm\frac{\Omega}{2}, \pm\frac{\Omega}{3}, \pm\frac{\Omega}{4} \quad . \tag{63}$$

We now proceed along the lines of section III. It is not necessary to write down all equations again, since almost all formal steps are identical. Therefore only those expressions are made explicit which change. We insert eq.(14) into the equation of motion, keeping in mind that the eigenvector is time–dependent, and compare order for order in ε.

To first order the Floquet–equation (61) is reproduced. The equation of motion for the second order is identical to eq.(30), where we again profit from the fact that k–independent eigenvectors can be assumed (cf. appendix B). Here iv of course denotes the derivative of the critical Floquet–exponent with respect to the wave number. For use of latter reference we write down the nonlinear contribution explicitly

$$
\begin{aligned}
(\mathcal{N}_2[\underline{\Phi}])^{[2]} &= 2\sum_{\alpha\beta}(iq_c)^\alpha(-iq_c)^\beta C_t^{(\alpha\beta)}\left\{\underline{u}_c(t),\underline{u}_c^*(t)\right\}|A|^2 \\
&+\sum_{\alpha\beta}(iq_c)^\alpha(iq_c)^\beta C_t^{(\alpha\beta)}\left\{\underline{u}_c(t),\underline{u}_c(t)\right\}e^{2iq_cx+2i\omega_ct}A^2 \\
&+\sum_{\alpha\beta}(-iq_c)^\alpha(-iq_c)^\beta C_t^{(\alpha\beta)}\left\{\underline{u}_c^*(t),\underline{u}_c^*(t)\right\}e^{-2iq_cx-2i\omega_ct}(A^*)^2 \quad , \qquad (64)
\end{aligned}
$$

where the T–periodic time–dependencies are indicated. Since either the wavenumber does not vanish or the non–resonance condition (63) holds, evaluation of the secular condition (B5) leads again to eq.(32). Hence the solution discarding transients is given by eq.(33) (cf. appendix B) but with $\underline{\Gamma}_a$ and $\underline{\Gamma}_b$ being replaced by the T–periodic quantities

$$
\begin{aligned}
\underline{\Gamma}_a(t) &:= \int_{-\infty}^{t}\underline{\underline{U}}_{k=0}(t,t')\sum_{\alpha\beta}(iq_c)^\alpha(-iq_c)^\beta C_{t'}^{(\alpha\beta)}\left\{\underline{u}_c(t'),\underline{u}_c^*(t')\right\}\,dt' \qquad (65)\\
&= \left[\underline{\underline{1}}-\underline{\underline{M}}_{k=0}\right]^{-1}\int_{t-T}^{t}\underline{\underline{U}}_{k=0}(t,t')\sum_{\alpha\beta}(iq_c)^\alpha(-iq_c)^\beta C_{t'}^{(\alpha\beta)}\left\{\underline{u}_c(t'),\underline{u}_c^*(t')\right\}\,dt' \\
\underline{\Gamma}_b(t) &:= \int_{-\infty}^{t}\underline{\underline{U}}_{2q_c}(t,t')e^{-2i\omega_c(t-t')}\sum_{\alpha\beta}(iq_c)^\alpha(iq_c)^\beta C_{t'}^{(\alpha\beta)}\left\{\underline{u}_c(t'),\underline{u}_c(t')\right\}\,dt' \qquad (66)\\
&= \left[\underline{\underline{1}}-\underline{\underline{M}}_{2q_c}e^{-2i\omega_cT}\right]^{-1}\int_{t-T}^{t}\underline{\underline{U}}_{2q_c}(t,t')e^{-2i\omega_c(t-t')}\sum_{\alpha\beta}(iq_c)^\alpha(iq_c)^\beta C_{t'}^{(\alpha\beta)}\left\{\underline{u}_c(t'),\underline{u}_c(t')\right\}\,dt' \quad .
\end{aligned}
$$

The indefinite time–integrals have been reduced to definite ones by using the Floquet–decomposition (B7) and the periodicity of the remaining factors.

We proceed to the third order and obtain eq.(41) with the derivatives of the eigenvalues being replaced by derivatives of the Floquet–exponents. For convenience we repeat the explicit expression for the nonlinear contributions

$$
\begin{aligned}
(\mathcal{N}_2[\underline{\Phi}])^{[3]} &= \underline{\Gamma}(t)e^{iq_cx+i\omega_ct}|A|^2A+\underline{\Gamma}^*(t)e^{-iq_cx-i\omega_ct}|A|^2A^*+\cdots \qquad (67)\\
(\mathcal{N}_3[\underline{\Phi}])^{[3]} &= \underline{\Delta}(t)e^{iq_cx+i\omega_ct}|A|^2A+\underline{\Delta}^*(t)e^{-iq_cx-i\omega_ct}|A|^2A^*+\cdots \quad . \qquad (68)
\end{aligned}
$$

Here $\underline{\Gamma}(t)$ and $\underline{\Delta}(t)$ are again given by eqs.(20) and (23) evaluated with the time dependent quantities. Transients and terms with wavenumber $\pm 2q_c, \pm 3q_c$ or frequency $\pm 2\omega_c, \pm 3\omega_c$, which do not contribute to the secular condition by virtue of the non–resonance condition (63), have been indicated by $\cdots$. The evaluation of the secular condition (B5) leads to eq.(42) and the subsequent considerations are the same as in section III. Hence we again obtain the amplitude equation (16). But since the integrand in the secular condition is time–dependent, the integral survives and the coefficients read

$$
\begin{aligned}
v &= \mathrm{Im}\left.\frac{d\mu^{(\nu_c)}(k)}{dk}\right|_{k=q_c} \qquad (69)\\
D &= -\frac{1}{2}\left.\frac{d^2\mu^{(\nu_c)}(k)}{dk^2}\right|_{k=q_c} \qquad (70)\\
r &= \frac{\int_0^T(\underline{v}_c(t)|\underline{\Gamma}(t)+\underline{\Delta}(t))\,dt}{\int_0^T(\underline{v}_c(t)|\underline{u}_c(t))\,dt} \qquad (71)\\
\eta &= \frac{q_c/(2\pi)\int_0^T\int_0^{2\pi/q_c}\left(\underline{v}_c(t)e^{iq_cx}|\mathcal{L}_t^{(2)}|\underline{u}_c(t)e^{iq_cx}\right)\,dx\,dt}{\int_0^T(\underline{v}_c(t)|\underline{u}_c(t))\,dt} \quad . \qquad (72)
\end{aligned}
$$

One might argue that the evaluation of these formulas is in general impossible. But one should notice that one only needs the solution of the time–dependent problem (B6) for one period (cf. eqs.(65) and (66)). This is at least numerically a very simple task, so that the evaluation even for quite complicated equations of motion can be performed on every computer. But also analytical computations are possible, if the Floquet–problem can be handled, e.g. with a separate perturbation expansion.

VII. STRONG RESONANCES

From the analysis of the preceding section it is obvious, that in cases of strong resonances additional terms contribute to the secular condition. We will discuss in the sequel the implications of each case separately and therefore assume $q_c = 0$ throughout this section.

a. Quartic Hopf–Hard–Mode Instability Consider the case, that the frequency of the marginally stable mode obeys[6]

$$\omega_c = \frac{\Omega}{4} \quad . \tag{73}$$

Since this condition puts an additional constraint on the instability, this situation can be roughly classified as a codimension–two instability. The analysis of the preceding section up to and including the equation of third order (cf. eq.(41)) is valid. But now the terms with frequency $\pm 3\omega_c$ also contribute to the secular condition. Hence for the evaluation of the nonlinear contribution (cf. eqs.(10), (12), (14), (33), (67), (68)) one has to considers these terms too

$$\begin{aligned}
(\mathcal{N}_2[\underline{\Phi}])^{[3]} &= \underline{\Gamma}(t)e^{i\omega_c t}|A|^2 A + \underline{\Gamma}^*(t)e^{-i\omega_c t}|A|^2 A^* \\
&+ \underline{\gamma}(t)\left(A^*\right)^3 e^{-3i\omega t + i\Omega t} + \underline{\gamma}^*(t) A^3 e^{3i\omega t - i\Omega t} + \cdots \qquad (74)\\
(\mathcal{N}_3[\underline{\Phi}])^{[3]} &= \underline{\Delta}(t)e^{i\omega_c t}|A|^2 A + \underline{\Delta}^*(t)e^{-i\omega_c t}|A|^2 A^* \\
&+ \underline{\delta}(t)\left(A^*\right)^3 e^{-3i\omega_c t + i\Omega t} + \underline{\delta}^*(t) A^3 e^{3i\omega_c t - i\Omega t} + \cdots \quad . \qquad (75)
\end{aligned}$$

Here

$$\begin{aligned}
\underline{\gamma}(t) &:= 2C_t^{(00)}\left\{\underline{u}_c^*(t), \underline{\Gamma}_b^*(t)\right\} e^{-i\Omega t} \qquad (76)\\
\underline{\delta}(t) &:= D_t^{(000)}\left\{\underline{u}_c^*(t), \underline{u}_c^*(t), \underline{u}_c^*(t)\right\} e^{-i\Omega t} \qquad (77)
\end{aligned}$$

denote the additional T–periodic coefficients[7], whereas $\cdots$ indicate those terms with frequency $\pm 2\omega_c$ which do not contribute to the secular condition. Because of the resonance condition (73) the last summands in eqs.(74) and (75) do not drop in eq.(B5). Instead of eq.(16) one obtains the amplitude equation, after having separated as usual the amplitudes A and B

$$\left(\frac{\partial}{\partial \tau_2} - v\frac{\partial}{\partial \xi_2}\right) A = \eta A + r|A|^2 A + s\left(A^*\right)^3 + D\frac{\partial^2 A}{\partial \xi_1^2} \quad . \tag{78}$$

The additional coefficient is given by

$$s = \frac{\int_0^T \left(\underline{v}_c(t)|\underline{\gamma}(t) + \underline{\delta}(t)\right)\, dt}{\int_0^T \left(\underline{v}_c(t)|\underline{u}_c(t)\right)\, dt} \quad , \tag{79}$$

whereas for the remaining quantities the formulas of the preceding section apply. In contrast to the usual hard–mode instability, the amplitude equation does not possess a phase symmetry. Hence the complex phase of η cannot be eliminated, and one has a two–dimensional "unfolding–parameter".

b. Flip–Hard–Mode Instability Let the frequency obey

$$\omega_c = \frac{\Omega}{2} \quad . \tag{80}$$

The corresponding Floquet–exponent is located at the boundary of the Brillouin zone, that means the Floquet–multiplier is isolated and takes the value -1. Such a value induces a period doubling bifurcation, which is of course a structurally stable bifurcation of codimension one, so that the condition (80) does not imply an additional constraint for the bifurcation. If one takes into account that $\underline{\underline{L}}(0,t)$ is a real matrix, then the complex part can be eliminated in the eigenvalue equations (61) and (62) using the abbreviations

---

[6] A pair of complex conjugate eigenvalues occurs $\omega^{(\nu_c)}(0) = \omega_c$ and $\omega^{(\nu_c')}(0) = -\omega_c$. The branch ν_c' leads to the complex conjugate expressions.

[7] Since $q_c = 0$ only the term $\alpha = \beta = \gamma = 0$ contributes in eqs.(20), (21), (22), and (23).

$$\underline{u}_c(t) =: e^{-i\omega_c t}\underline{\hat{u}}_c(t), \quad \underline{\hat{u}}_c(t) = -\underline{\hat{u}}_c(t+T) \in \mathbb{R}^N \tag{81}$$
$$\underline{v}_c^*(t) =: e^{i\omega_c t}\underline{\hat{v}}_c(t), \quad \underline{\hat{v}}_c(t) = -\underline{\hat{v}}_c(t+T) \in \mathbb{R}^N \quad . \tag{82}$$

Here the real vector $\underline{\hat{u}}_c(t)$ points into the direction of the centre manifold, which in this case is a Möbius strip. As a consequence eq.(14) simplifies to

$$\begin{aligned} \underline{\Phi}(x,t) &= 2\varepsilon\underline{\hat{u}}_c(t)A_r(\tau_1,\tau_2,\ldots,\xi_1,\xi_2,\ldots) + \varepsilon^2\underline{\Phi}^{(2)} + \varepsilon^3\underline{\Phi}^{(3)} + \cdots \\ A_r &:= \mathrm{Re}A \quad , \end{aligned} \tag{83}$$

so that the field is completely determined by real part of the amplitude. This property is a direct consequence of the fact, that the centre manifold in the homogeneous system is a one–dimensional real manifold. It is now quite simple to evaluate the consequences of eqs.(80), (81), and (82) for the amplitude equation. To second order one has the result (30) and (64). But to the secular condition (B5) now both, the second and third summand on the right hand side of eq.(30) contribute

$$0 = \left(v\frac{\partial}{\partial\xi_1} - \frac{\partial}{\partial\tau_1}\right)(A + A^*) = 2\left(v\frac{\partial}{\partial\xi_1} - \frac{\partial}{\partial\tau_1}\right)A_r \quad . \tag{84}$$

In addition, if we insert eqs.(81) and (82) into the definitions (65) and (66) we have

$$\underline{\Gamma}_a(t) = e^{2i\omega_c t}\underline{\Gamma}_b(t) = \int_{-\infty}^{t} \underline{\underline{U}}_{k=0}(t,t')C_{t'}^{(00)}\left\{\underline{\hat{u}}_c(t'),\underline{\hat{u}}_c(t')\right\}\,dt' \in \mathbb{R}^N \quad , \tag{85}$$

since the evolution matrix is a real quantity. Hence the solution (33) of second order simplifies to

$$\underline{\Phi}^{(2)} = \underline{\Gamma}_a(t)\,(A+A^*)^2 + \underline{\hat{u}}_c(t)\,(B+B^*) = 4\underline{\Gamma}_a(t)A_r^2 + 2\underline{\hat{u}}_c(t)B_r \quad . \tag{86}$$

To third order we already have eq.(41), where all the cubic terms in the amplitude contribute, that means (74) and (75) apply. If we evaluate the coefficients (20), (23), (76), and (77) using eqs.(80), (81), (82), and (85) we obtain

$$(\mathcal{N}_2[\underline{\Phi},t])^{[3]} = \underline{\hat{\Gamma}}(t)\,(A+A^*)^3 + \cdots \tag{87}$$
$$(\mathcal{N}_3[\underline{\Phi},t])^{[3]} = \underline{\hat{\Delta}}(t)\,(A+A^*)^3 + \cdots \tag{88}$$

where

$$\underline{\hat{\Gamma}}(t) = \frac{1}{3}e^{i\omega_c t}\underline{\Gamma}(t) = 2C_t^{(00)}\left\{\underline{\hat{u}}_c(t),\underline{\Gamma}_a(t)\right\} \in \mathbb{R}^N \tag{89}$$
$$\underline{\hat{\Delta}}(t) = \frac{1}{3}e^{i\omega_c t}\underline{\Delta}(t) = D_t^{(000)}\left\{\underline{\hat{u}}_c(t),\underline{\hat{u}}_c(t),\underline{\hat{u}}_c(t)\right\} \in \mathbb{R}^N \quad . \tag{90}$$

The secular condition (B5) then leads to the amplitude equation

$$\left(\frac{\partial}{\partial\tau_2} - v\frac{\partial}{\partial\xi_2}\right)A_r = \eta A_r + 4rA_r^3 + D\frac{\partial^2 A_r}{\partial\xi_1^2} \quad , \tag{91}$$

if the amplitude B_r is separated as usual. Again the velocity and the real diffusion coefficient are determined by the derivatives of the spectrum (69) and (70), whereas the remaining quantities are expressed in terms of the centre manifold coordinate

$$r = \frac{\int_0^T \left(\underline{\hat{v}}_c(t)|\underline{\hat{\Gamma}}(t) + \underline{\hat{\Delta}}(t)\right)\,dt}{\int_0^T (\underline{\hat{v}}_c(t)|\underline{\hat{u}}_c(t))\,dt} \in \mathbb{R} \tag{92}$$
$$\eta = \frac{q_c/(2\pi)\int_0^T\int_0^{2\pi/q_c}\left(\underline{\hat{v}}_c(t)|\mathcal{L}_t^{(2)}|\underline{\hat{u}}_c(t)\right)\,dx\,dt}{\int_0^T (\underline{\hat{v}}_c(t)|\underline{\hat{u}}_c(t))\,dt} \in \mathbb{R} \quad . \tag{93}$$

Eq.(91), which is entirely real including the amplitude, should not be mixed up with the real Ginzburg–Landau equation. It is sometimes called a Fishers equation [14].

c. Cubic Hopf–Hard–Mode Instability Yet there appeared only modifications in the amplitude equations, but the general perturbation scheme was not influenced. This feature changes considerably for a third order degeneracy in the Floquet–multipliers, i.e.

$$\omega_c = \frac{\Omega}{3} \quad . \tag{94}$$

It is evident from eqs.(30) and (64) that the nonlinearities contribute to the secular condition. Indeed we obtain[8]

$$\left(\frac{\partial}{\partial \tau_1} - v\frac{\partial}{\partial \xi_1}\right) A = \alpha\,(A^*)^2 \quad , \tag{95}$$

with a coefficient α being determined by quadratic nonlinearities. It is not difficult to show, that for almost all initial conditions eq.(95) yields an unbounded, that means a secular, solution. This feature is by no means amazing, since a stabilising cubic term, known to be important in the normal form of the spatially homogeneous system, is missing. One may cure this defect by tracing back to what is sometimes called a re-summation of the secular conditions [5, p.318f]. Although such an approach seems to be quite common, it is difficult to estimate the validity of this procedure. Especially the actual expansion parameter is unclear and one mixes the different scales in an uncontrolled manner. In fact, the formal concept of the multiple scale analysis implies, that the secular conditions at each order have to be satisfied separately. In this sense the codimension–two bifurcation of this paragraph cannot be treated by the perturbation scheme.

VIII. CONCLUSION

It was shown by explicit calculation that for every codimension–one instability of a trivial state the dynamics beyond the threshold is governed by a Ginzburg–Landau or a Fishers equation. Especially for the frequently met case of a soft–mode or a hard–mode instability in autonomous systems, there arises a unique expression for the coefficients of the amplitude equation, which covers both the real as well as the complex Ginzburg–Landau equation. The reader may object that all these results can be obtained simply from normal forms of ordinary differential equations supplemented with symmetry considerations, and that its is not necessary to go through the explicit derivation. Alongside the disadvantage, that such an approach is incapable to yield numerical values for the coefficients, e.g. to locate transitions from super– to sub–critical instabilities, one has to be careful concerning the validity of such phenomenological amplitude equations. It might happen, that in the stage of the derivation secular conditions occur, which put severe constraints on the validity of the multiple scales approach. One such constraint, which in fact invalidates the approach, was presented in section VII in conjunction with strong resonances. A more prominent example is known in the context of counter–propagating waves [15]. Here the thorough approach leads to a nonlocal reduced description, whereas the validity of the phenomenological amplitude equations containing convection terms is limited to a higher–codimension instability. A similar phenomenon is known in the context of phase turbulence, described by the Kuramoto–Sivashinski equation [16]. Hence symmetry considerations and normal forms of ordinary differential equations are very good guidelines for the resulting amplitude equation, but they do not substitute a formal derivation.

The general and formal approach has shown that it is by no means essential to implement certain properties of the perturbation expansion with ad hoc assumptions. In fact all scales which seem to be superfluous drop by itself, which emphasises the consistency of the perturbation expansion. One should however stress, that the whole physics (or mathematics, depending on the readers taste), is contained in the expansion (14) of the solution. All the remaining steps are just straightforward. Hence, by a change of this expansion it is obvious, that different situations, e.g. higher–codimension instabilities, phase equations etc. can be also handled for a general equation of motion. Since the whole formalism is quite simple, it is indeed possible to treat higher–codimension bifurcations, and to obtain the amplitude equation from the basic equation of motion as demonstrated on the example of the Turing–Hopf instability. Different cases, e.g. the degenerated soft–mode instability which require perturbation expansions of higher order have been already treated [17], and will be published elsewhere.

Finally the formal and general nature of the presented treatment may clearly indicate the principal limits of the approach by amplitude equations. Although it was not our purpose to touch the asymptotic properties of the

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[8] A linear and a diffusive term can be introduced by using a different scaling of the amplitude and the spatial coordinate with ε. It does not seem to change the subsequent considerations.

expansion from the mathematical point of view, our scheme probably contributes to this field as well, e.g. one may construct a clear connection to the normal form theory of low dimensional dynamical systems. But a thorough investigation will require methods which are beyond the scope of this publication.

APPENDIX A: TRANSFORMATION PROPERTIES

Suppose $\underline{u}_k^{(\nu_c)}$ depends on k. We can perform a unitary transformation $\underline{\underline{R}}(k)$ depending continuously on k, so that

$$\underline{\underline{R}}(k)\underline{u}_k^{(\nu_c)} = \underline{u}_c, \quad \underline{\underline{R}}(q_c) = \underline{\underline{1}} \tag{A1}$$

holds. The symmetry relation $\underline{\underline{R}}^*(k) = \underline{\underline{R}}(-k)$ may be imposed[9]. We are now going to use this rotation to transform the full partial differential equation (1) in order to obtain k–independent critical eigenvectors. The main trick however is, that the final results (cf. section II) are scalars so that the transformation cancels in these formulas. Hence the evaluation of these expressions based on the original equation is permitted and one may not take any notice from the transformation.

To perform the transformation (A1) on each Fourier–mode let us define the real operator

$$\mathcal{R} := \underline{\underline{R}}\left(-i\frac{\partial}{\partial x}\right) \quad . \tag{A2}$$

Its inverse can also been introduced since $\underline{\underline{R}}(k)$ is unitary. With the transformed field

$$\tilde{\underline{\Phi}}(x,t) := \mathcal{R}\underline{\Phi}(x,t) \tag{A3}$$

the equation of motion reads

$$\frac{\partial\tilde{\underline{\Phi}}}{\partial t} = \mathcal{R}\mathcal{L}\mathcal{R}^{-1}\tilde{\underline{\Phi}} + \mathcal{R}\mathcal{N}[\mathcal{R}^{-1}\tilde{\underline{\Phi}}] =: \tilde{\mathcal{L}}\tilde{\underline{\Phi}} + \tilde{\mathcal{N}}[\tilde{\underline{\Phi}}] \quad . \tag{A4}$$

We now apply the derivation of section III, since by definition (A1) the eigenvector $\tilde{\underline{u}}_k^{(\nu_c)}$ of

$$\tilde{\underline{\underline{L}}}(k) := \underline{\underline{R}}(k)\underline{\underline{L}}(k)\underline{\underline{R}}^{-1}(k) \tag{A5}$$

does not depend on k. Hence we obtain the result of section II but of course the coefficients (17), (18), (19), and (24) are expressed in terms of the new quantities. We have to show that the transformation drops from this expression.

First the eigenvalues $\lambda^{(\nu)}(k)$ are independent of the transformation $\mathcal{R}$, so that expression (17) and (18) can be evaluated in terms of the original quantities (1). Second the transformed field (A3) has an amplitude $\tilde{A}$ which may differ from the amplitude A of the original field. But if we insert the definition (14) into eq.(A3), we clearly observe, taking eq.(A1) into account, that both amplitudes coincide up to order ε. Third the transformation drops from the matrix element (24), if the definition for $\tilde{\mathcal{L}}$ (cf. eqs.(A3) and (A4)) is inserted. Finally we have to show, that the nonlinear coefficient can be evaluated from the original quantities.

For that purpose consider the transformed nonlinearity $\tilde{\mathcal{N}}_2$, insert a field consisting of two Fourier–modes $\tilde{\underline{\psi}} = \tilde{\underline{\psi}}_{q_1}\exp(iq_1x) + \tilde{\underline{\psi}}_{q_2}\exp(iq_2x) + c.c.$, and equate the Fourier–component $\exp(i(q_1+q_2)x)$. Then, according to the definition (10) and the transformation (A4) one obtains

$$\sum_{\alpha\beta}(iq_1)^\alpha(iq_2)^\beta\tilde{C}^{(\alpha\beta)}\left\{\tilde{\underline{\psi}}_{q_1},\tilde{\underline{\psi}}_{q_2}\right\} = \sum_{\alpha\beta}(iq_1)^\alpha(iq_2)^\beta\underline{\underline{R}}(q_1+q_2)C^{(\alpha\beta)}\left\{\underline{\underline{R}}^{-1}(q_1)\tilde{\underline{\psi}}_{q_1},\underline{\underline{R}}^{-1}(q_2)\tilde{\underline{\psi}}_{q_2}\right\} \quad . \tag{A6}$$

This formula tells us how the tensors change under the transformation. We now repeatedly use this relation to evaluate the nonlinear coefficient by choosing q_1, q_2, $\tilde{\underline{\psi}}_{q_1}$, and $\tilde{\underline{\psi}}_{q_2}$ appropriately. Consider the definition of $\tilde{\underline{\Gamma}}_a$ (cf. eq.(21)) and apply relation (A6). Then

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[9]We need the transformation as a formal tool only in a neighbourhood of $|k| = |q_c|$. Hence we need not worry about the global continuation.

$$\underline{\tilde{\Gamma}}_a = -\frac{1}{\underline{\underline{\tilde{L}}}(0)}\underline{\underline{R}}(0)\sum_{\alpha\beta}(iq_c)^\alpha(-iq_c)^\beta C^{(\alpha\beta)}\left\{\underline{\underline{R}}^{-1}(q_c)\underline{\tilde{u}}_c, \underline{\underline{R}}^{-1}(-q_c)\underline{\tilde{u}}_c^*\right\} = \underline{\underline{R}}(0)\underline{\Gamma}_a \tag{A7}$$

holds, where eqs.(A1) and (A5) were used in the last step. In the same way one obtains

$$\underline{\tilde{\Gamma}}_b = \underline{\underline{R}}(2q_c)\underline{\Gamma}_b \quad . \tag{A8}$$

If we now insert both expression into the definition (20) and use again the transformation (A6) appropriately we end up with

$$\begin{aligned}\underline{\tilde{\Gamma}} &= 2\sum_\alpha (iq_c)^\alpha \underline{\underline{R}}(q_c)C^{(\alpha 0)}\left\{\underline{\underline{R}}^{-1}(q_c)\underline{\tilde{u}}_c, 2\underline{\underline{R}}^{-1}(0)\underline{\tilde{\Gamma}}_a\right\} + 2\sum_{\alpha\beta}(2iq_c)^\beta(-iq_c)^\alpha \underline{\underline{R}}(q_c)C^{(\alpha\beta)}\left\{\underline{\underline{R}}^{-1}(-q_c)\underline{\tilde{u}}_c^*, \underline{\underline{R}}^{-1}(2q_c)\underline{\tilde{\Gamma}}_b\right\} \\ &= \underline{\Gamma} \quad .\end{aligned} \tag{A9}$$

In the same way one obtains

$$\underline{\tilde{\Delta}} = \underline{\Delta} \quad , \tag{A10}$$

so that the nonlinear coefficient r can be evaluated from the transformed as well as the original equation using the definitions (19), (20), (21), (22), and (23) in terms of the original quantities (1)

The same approach can be used in the explicitly time–dependent case as well. In that case one chooses a T–periodic transformation to make the Floquet–vectors k–independent

$$\underline{\underline{R}}(k,t)\underline{u}_k^{(\nu_c)}(t) = \underline{u}_c(t), \quad \underline{\underline{R}}(q_c,t) = \underline{\underline{1}} \quad . \tag{A11}$$

The only difference to the preceding considerations results from the time–dependence, which contributes an additional term to the transformed linear operator (cf. eq.(A4))

$$\tilde{\mathcal{L}}_t = \mathcal{R}_t\mathcal{L}_t\mathcal{R}_t^{-1} + \dot{\mathcal{R}}_t\mathcal{R}_t^{-1} \quad . \tag{A12}$$

But it is exactly this property which ensures, that the propagators (B6), occurring in the definitions (65) and (66) of $\underline{\tilde{\Gamma}}_a(t)$ and $\underline{\tilde{\Gamma}}_b(t)$ respectively, transform according to

$$\underline{\underline{\tilde{U}}}_k(t,t') = \underline{\underline{R}}(k,t)\underline{\underline{U}}_k(t,t')\underline{\underline{R}}(k,t') \quad . \tag{A13}$$

Hence the transformation properties (A7), (A8), (A9), and (A10) are valid in the time–dependent case too.

APPENDIX B: SECULAR CONDITION

Let us determine the solution of the linear equation

$$\frac{\partial \underline{\psi}}{\partial t} = \mathcal{L}^{(0)}\underline{\psi} + \sum_k \underline{w}_k(t)e^{ikx} \quad , \tag{B1}$$

presupposing that the linear operator has only stable modes except for one (cf. section II) and that the inhomogeneous part, bounded in time, is given by a finite sum of Fourier–modes.

By considering each Fourier–component separately, the partial differential equation reduces to ordinary differential equations and the general solution is easily written down in terms of the matrix (6)

$$\underline{\psi}(x,t) = \underline{u}_c e^{iq_cx+i\omega_ct}B + \sum_k \int_0^t \exp\left[\underline{\underline{L}}(k)t'\right]\underline{w}_k(t-t')\,dt'e^{ikx} + \cdots \quad . \tag{B2}$$

Here B denotes the constant of integration, and $\cdots$ that part of the homogeneous solution, which corresponds to stable eigenvalues and leads to a transient only. Consider first the non–critical summands, that means $|k| \neq |q_c|$. Then the integrals converge in the long time limit, since all eigenvalues have a negative real part. For the marginally stable wavenumber $k = q_c$ one eigenvalue with a vanishing real part occurs, so that the integral may increase in time. By using e.g. the spectral decomposition of the corresponding matrix

$$\underline{\underline{L}}(q_c) = \sum_\mu \lambda^{(\mu)}(q_c) \frac{|\underline{u}_{q_c}^{(\mu)})(\underline{v}_{q_c}^{(\mu)}|}{(\underline{v}_{q_c}^{(\mu)}|\underline{u}_{q_c}^{(\mu)})} \quad , \tag{B3}$$

it is evident from eq.(B2) that a secular contribution increasing linearly in time is avoided if the relation

$$0 = \lim_{\Theta\to\infty} \frac{1}{\Theta} \int_0^\Theta e^{-i\omega_c t'} (\underline{v}_c|\underline{w}_{q_c}(t'))\, dt' \tag{B4}$$

is fulfilled[10]. In this case the solution in the stationary state is obtained by extending the upper limit of the time integrals in eq.(B2) to infinity.

For a T–periodic explicitly time–dependent operator $\mathcal{L}_t^0$ similar considerations apply. The secular condition is e.g. obtained from eq.(B1) by multiplication with the critical left–Floquet–eigenfunction $\underline{v}_c(t)e^{iq_c x}$ (cf. eq.(62)). One has to require

$$0 = \lim_{\Theta\to\infty} \frac{1}{\Theta} \int_0^\Theta e^{-i\omega_c t'} (\underline{v}_c(t')|\underline{w}_{q_c}(t'))\, dt' \tag{B5}$$

in order to exclude linearly in time increasing contributions. The solution of eq.(B1) is again determined by the expression (B2) with the matrix–exponential being replaced by the corresponding evolution operator $\underline{\underline{U}}_k(t, t-t')$ of the time dependent system. The latter is determined by

$$\frac{\partial \underline{\underline{U}}_k(t,t')}{\partial t} = \underline{\underline{L}}(k,t)\underline{\underline{U}}_k(t,t'), \quad \underline{\underline{U}}_k(t',t') = \underline{\underline{1}} \quad . \tag{B6}$$

The solution in the stationary state is again obtained by extending the time integrals to infinity. This indefinite integrals can be reduced to definite ones by taking the Floquet–decomposition

$$\underline{\underline{U}}_k(t+T,t') = \underline{\underline{M}}_k \underline{\underline{U}}_k(t,t') \tag{B7}$$

into account. Here the constant matrix $\underline{\underline{M}}_k$ determines the Floquet–multipliers.

APPENDIX C: MAXWELL–BLOCH EQUATIONS

For purely pedagogical purpose we present the evaluation of the expressions given in section II for the Laser instability, which is governed by the Maxwell–Bloch equations [18]. Thus every formal step is performed explicitly. For the physical details however the reader should consult the literature.

The basics equations of motion, which govern the evolution of the complex valued envelopes e and p of the electric and the polarisation field as well as the deviation of the inversion n from the pump level r, read in dimensionless form

$$\begin{aligned} \frac{\partial e}{\partial t} &= ia\frac{\partial^2 e}{\partial x^2} - \sigma(e-p) \\ \frac{\partial p}{\partial t} &= -(1+i\Omega_0)p + (r-n)e \\ \frac{\partial n}{\partial t} &= -bn + (e^*p + ep^*)/2 \end{aligned} \tag{C1}$$

where x denotes the direction perpendicular to the beam. Here the parameters σ, Ω_0, and b denote the cavity damping, the detuning, and the decay rate of the inversion in units of the dephasing rate. The parameter a scales the diffraction term. In the sequel we consider $\Omega_0 < 0$ and analyse the instability of the trivial state $e = p = n = 0$. Introducing the five real quantities via

$$e = e^{-i\Omega_0 t}(\Phi_1 + i\Phi_2), \quad p = e^{-i\Omega_0 t}(\Phi_3 + i\Phi_4), \quad n = \Phi_5 \tag{C2}$$

[10] Often such a relation is called a solvability condition for the linear eq.(B1). But such a term requires the specification of function spaces on which the operators are considered.

the equation of motion is cast into the form (1)

$$\frac{\partial \underline{\Phi}}{\partial t} = \begin{pmatrix} -\sigma & -\Omega_0 - a\partial_x^2 & \sigma & 0 & 0 \\ \Omega_0 + a\partial_x^2 & -\sigma & 0 & \sigma & 0 \\ r & 0 & -1 & 0 & 0 \\ 0 & r & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -b \end{pmatrix} \underline{\Phi} + \begin{pmatrix} 0 \\ 0 \\ -\Phi_1\Phi_5 \\ -\Phi_2\Phi_5 \\ \Phi_1\Phi_3 + \Phi_2\Phi_4 \end{pmatrix} \quad . \tag{C3}$$

Since the nonlinearity is quadratic, the only nonvanishing contribution using the notation (9), (10) and respecting the symmetry constraint reads

$$\begin{aligned} C^{(00)}\{\underline{u}, \underline{v}\} = \frac{1}{2}\,&(0, 0, -u_1v_5 - u_5v_1, -u_2v_5 - u_5v_2, \\ &u_1v_3 + u_3v_1 + u_2v_4 + u_4v_2)^T \quad . \end{aligned} \tag{C4}$$

The matrix (6) is obtained from eq.(C3) by replacing the differential ∂_x by ik. Its characteristic polynomial reads

$$0 = [\lambda + b]\left[\Omega_k^2 + \sigma^2(1-r)^2 + 2\lambda\{\Omega_k^2 + (1-r)(1+\sigma)\sigma\} + \lambda^2\{(1+\sigma)^2 + \Omega_k^2 + 2\sigma(1-r)\} + 2\lambda^3\{1+\sigma\} + \lambda^4\right] \tag{C5}$$

with the abbreviation $\Omega_k := \Omega_0 - ak^2$. To determine the instability threshold $r = r_c$ one inserts $\lambda = i\omega_c$ and obtains, by separating the real and imaginary part, the critical frequency

$$\omega_c^2 = \frac{\Omega_{q_c}^2 + (1-r_c)(1+\sigma)\sigma}{1+\sigma} \tag{C6}$$

and the threshold

$$r_c = 1 + \left(\frac{\Omega_{q_c=0}}{1+\sigma}\right)^2 = 1 + \omega_c^2 \quad . \tag{C7}$$

The instability occurs at $q_c = 0$ since the threshold (C7) is minimal for that wavenumber. The right and left eigenvectors (cf. eqs.(5) and (8)) corresponding to the eigenvalue $i\omega_c := i\Omega_0/(1+\sigma)$ can be read off from the matrix as

$$\underline{u}_c = (i, 1, i(1-i\omega_c), 1-i\omega_c, 0)^T \,, \quad \underline{v}_c^* = (1, i, \frac{\sigma}{1+i\omega_c}, i\frac{\sigma}{1+i\omega_c}, 0) \quad . \tag{C8}$$

Introducing the deviation from the threshold by $r = r_c + \varepsilon^2\delta$, the perturbation $\mathcal{L}^{(2)}$ has only two nonvanishing matrix elements and eq.(24) yields

$$\eta = \delta\frac{(\underline{v}_c^*)_3(\underline{u}_c)_1 + (\underline{v}_c^*)_4(\underline{u}_c)_2}{(\underline{v}_c|\underline{u}_c)} = \delta\frac{\sigma}{1+\sigma+i\omega_c(1-\sigma)} \quad . \tag{C9}$$

Keeping in mind, that $\underline{\underline{L}}(k)$ depends solely on k^2, it is obvious that the convective velocity v vanishes. The second derivative, i.e. the diffusion constant, is easily expressed in terms of the eigenvectors (C8) by tracing back to the usual Schrödinger perturbation expansion

$$D = -a\frac{(\underline{v}_c^*)_1(\underline{u}_c)_2 - (\underline{v}_c^*)_2(\underline{u}_c)_1}{(\underline{v}_c|\underline{u}_c)} = ia\frac{1+i\omega_c}{1+\sigma+i\omega_c(1-\sigma)} \quad . \tag{C10}$$

Because of eqs.(C4) and (C8) $C^{(00)}\{\underline{u}_c, \underline{u}_c\}$ vanishes identically, and $C^{(00)}\{\underline{u}_c, \underline{u}_c^*\}$ has only one nonvanishing component

$$C^{(00)}\{\underline{u}_c, \underline{u}_c^*\} = (0, 0, 0, 0, 2)^T \quad . \tag{C11}$$

Then by virtue of eqs.(21) and (22) we have $\underline{\Gamma}_b = 0$ and

$$\underline{\Gamma}_a = -\frac{1}{\underline{\underline{L}}(0)}C^{(00)}\{\underline{u}_c, \underline{u}_c^*\} = \frac{2}{b}(0, 0, 0, 0, 1)^T \quad , \tag{C12}$$

so that eq.(20) reads

$$\underline{\Gamma} = 2C^{(00)}\{\underline{u}_c, 2\underline{\Gamma}_a\} = \frac{4}{b}\,(0,0,-(\underline{u}_c)_1, -(\underline{u}_c)_2, 0)^T \quad . \tag{C13}$$

Since $\underline{\Delta} = 0$ because of the absence of cubic nonlinearities we obtain finally for the nonlinear coefficient (19)

$$r = \frac{4}{b}\frac{-(\underline{v}_c^*)_3(\underline{u}_c)_1 - (\underline{v}_c^*)_4(\underline{u}_c)_2}{(\underline{v}_c|\underline{u}_c)} = -\frac{4}{b}\frac{\sigma}{1+\sigma+i\omega_c(1-\sigma)} \quad . \tag{C14}$$

The formal simplicity of this example suggests, that the explicit evaluation is not at all a difficult task, even in cases which are usually believed to involve lengthy calculations like the Rayleigh–Bénard problem.

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