Time Correlations of Intermittent Maps in the Non–Hyperbolic Phase

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Abstract

Time correlations of type I intermittent systems are investigated in terms of the thermodynamical approach. Especially the influence of the oder of tangency in the one dimensional model map on the order–q power spectrum is discussed.

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1 Introduction

Among the phenomena emerging in chaotic nonlinear dynamical systems Intermittency is probably the most striking one. Since the discovery of Pomeau and Manneville [1] and Grebogi et. al. [2] it has been discovered in a variety of physical systems. In this context type I–III intermittency are probably the most popular ones which can be modelled theoretically by one dimensional maps. The time evolution is governed by long laminar motions which are interrupted by turbulent bursts. This situation yields a special structure for the time correlations and leads to a typical form of the power spectrum in the low frequency region. It is well known that the tangency in the associated one dimensional map results in a power law decay of the power spectrum where the exponent is determined by the order of the tangency [3]. Additionally in the case of type I intermittency which is governed by a saddle node bifurcation the nearly periodic laminar phases induce a fine line structure which is determined by the inverse of the mean laminar period [4]. It is however an open question whether the approach by double time correlations is sufficient to characterize intermittency. This paper outlines several points in this direction especially by focusing on simple type I intermittent systems.

In order to characterize the ergodic properties of chaotic systems not only the physical (SRB) measure and its correlations are importatnt but also the different invariant sets contain information concerning the structure of the chaotic set. To analyse these local structures several approaches have been developed in the context of the thermodynamical formalism of dynamical systems [5]. One of this tool is the so called order–q power spectrum defined via [6]

$$I_q(\omega) = \lim_{n\to\infty} \langle \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} u(T^k(x)) e^{-i\omega k} \right|^2 \exp(qnU_n(x))\rangle / \langle \exp(qnU_n(x))\rangle \qquad (1)$$

with respect to a dynamical system $x_{n+1} = T(x_n)$ and an observable $u(x)$[1]. Here $U_n(x) := \sum_{k=0}^{n-1} u(T^k(x))/n$ denotes a finite time average. For $q = 0$ this quantity obviously coincides with the ordinary power spectrum whereas for $q \neq 0$ different characteristics of the variable $u(x)$ can be detected. It has been shown recently that they are generated by different invariant sets of the dynamical system so that by changing q different local structures of the underlying attractor can be detected [7]. Especially in the case of intermittent systems this results in the fact that the quantity (1) shows a non–analytic behaviour at a certain value q_c usually called a phase transition point [7, 8, 9]. The behaviour for $q < q_c$ ($q > q_c$) is typically determined soley by the laminar (turbulent) part of the motion. We are here concerned with the laminar phase only as the turbulent phase depends strongly on the nature of the underlying system in contrast to the former one. Our investigations are motivated

[1] $\langle \cdots \rangle$ denotes a long time or Sinai–Ruelle–Bowen average.

by recent investigations on the intermittency in a coupled electronic circuit [10]. The evaluation of the experimental data shows a power law decay of the spectrum (1) whereas the exponent depends on the parameter q. In this letter we want to analyse whether such a behaviour has some relations to the order of tangency in type one intermittent systems which governs the behaviour of the ordinary power spectrum.

Concerning the theoretical approach towards the quantity (1) the transfer operator [11]

$$(\mathcal{H}_q^u h)(x) = \int \delta(x - T(y)) \exp(qu(y)) h(y) dy \tag{2}$$

plays a central role. Its eigenvalue spectrum determines completly the quantity (1) [6]. In terms of the largest eigenvalue λ_q its right eigenfunction $h(x)$ and its left eigenmeasure ν[2] the following explicit relation has been derived recently [7]

$$I_q(\omega) = \int u(x) \left\{ (1 - \frac{e^{i\omega}}{\lambda_q} \mathcal{H}_q^u)^{-1} + (1 - \frac{e^{-i\omega}}{\lambda_q} \mathcal{H}_q^u)^{-1} - 1 \right\} (u \cdot h)(x) d\nu \tag{3}$$

where the normalization $\int h(x)\, d\nu = 1$ has been imposed. In general it is however difficult to diagonalize the operator (2) and to evaluate the expression (3) explicitly. We will therefore treat in the sequel one dimensional type I intermittent model maps where the evaluation in the vicinity of the bifurcation point is possible. Special emphasis is put on the relation between the order of the tangency and the asymptotic shape of the spectrum for small frequencies.

2 Model for type I intermittency

In order to deal with the type I intermittent dynamics we refer back to the associated one dimensional map which undergoes a saddle node bifurcation. In approaching the bifurcation point the map developes a tangency of the form $x + Ax^\zeta$ where the order $\zeta > 1$ of the tangency is the important parameter in our investigations. To deal with the problem almost analytically we approximate the full map by a piecewise linear function which allows for a finite dimensional matrix representation of the transfer operator (2) and hence for an explicit evaluation of the power spectrum (3) [8, 7]. We give a brief survey of our model in the sequel but refer to [7] for the explicit calculations.

As the system approaches the bifurcation point it passes unstable periodic orbits of increasing length (cf. Fig.1) where the length N can serve as the bifurcation parameter. By this property a Markov partition $\{I_k^{(N)}\}$ is generated. We replace the map on the sets of this partition by a linear function which is an arbitrary good approximation in the vicinity of the bifurcation point. Although this construction fails < Fig.1

[2] $\forall g,\ \int \mathcal{H}_q^u g(x)\, d\nu = \lambda_q \int g(x)\, d\nu$

to be a reasonable approximation in the turbulent region $I_{N+1}^{(N)}$ it induces neglegible errors as we are here concerned with the laminar phase only. By this reasoning we obtain the map depicted in Fig.1 where for simplicity of the computation a symmetry with respect to the off diagonal has been imposed [7]. The shape of the map is determined by the width of the intervals $I_k^{(N)}$, $\Delta a_k^{(N)}$, $-N \le k \le N+1$. To achieve the dependence on the bifurcation parameter N in a uniform way we require that

$$\begin{aligned} \Delta a_{N+1-k}^{(N)} = \Delta a_{-N+k}^{(N)} = c_k \alpha^{(N)}, \quad 0 \le k \le N \\ 1 = \sum_{k=1}^{N+1} \Delta a_k^{(N)} = \alpha^{(N)} \sum_{k=0}^{N} c_k \end{aligned} \tag{4}$$

where c_k denotes a fixed N independent sequence with limit zero. It is checked easily that the order of the tangency ζ defined above and the asymptotic behaviour of this sequence $c_k \sim k^{-a}$ are related via $a = \zeta/(\zeta-1)$. The case of a quadratic tangency corresponds to the value $a = 2$ [12, 7] whereas the So–Ose–Mori map [13] is obtained in the limit $a \to \infty$. For our observable $u(x)$ we make the simplest choice which reflects the basic features of the intermittent dynamics, namely a function taking values 0 and 1 in the laminar respectively turbulent regions

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in \bigcup_{k=-N}^{N} I_k^{(N)} \\ 1 & \text{if} \quad x \in I_{N+1}^{(N)} \end{cases} \quad . \tag{5}$$

The results are almost independent of this choice but we refer the reader to [14] for the explicit discussion of this topic. With these settings the transfer operator admits a finite dimensional matrix representation. With its help the following expression for the power spectrum can be derived from eq.(3) [7]

$$I_q(\omega) = \mu_{N+1} \left(2\mathrm{Re} \frac{1}{Z_N(e^{i\omega}/\lambda_q)} - 1 \right) \tag{6}$$

where

$$\begin{aligned} Z_N(z) &:= 1 - e^{q-q_c} \left\{ z^{2N+3} g_N\left(\frac{1}{z}\right) + g_N(z) \right\} \\ g_N(z) &:= \alpha^{(N)} \sum_{k=0}^{N} c_k z^{k+1} \end{aligned} \tag{7}$$

and the phase transition point is given by $q_c = \ln 2$. The eigenvalue λ_q and the normalization constant μ_{N+1} are determined by the equations

$$\begin{aligned} 0 &= Z_N\left(\frac{1}{\lambda_q}\right) \\ 0 &= 1 + \frac{1}{\lambda_q} Z_N'\left(\frac{1}{\lambda_q}\right) \mu_{N+1} \quad . \end{aligned} \tag{8}$$

In the vicinity of the bifurcation point, $N \to \infty$, on which we will concentrate the subsequent discussion, the eigenvalue approaches 1 in the laminar phase, $\lambda_q = 1 + O(1/N)$, $q < q_c$. Then the asymptotic expression for the power spectrum reads

$$I_q(\omega) \simeq \mu_{N+1} \left\{ 2\mathrm{Re}\frac{1}{1 - e^{q-q_c}\left(e^{i(2N+3)\omega} g(e^{-i\omega}) + g(e^{i\omega})\right)} - 1 \right\} \tag{9}$$

where

$$g(z) = \alpha^{(\infty)} \sum_{k=0}^{\infty} c_k z^{k+1} \tag{10}$$

denotes the $N \to \infty$ limit of eq.(7_2). The latter quantity contains the whole information about the shape of the map and will play a key role in determing the influence of the order of tangency on the power spectrum. Due to the rapid oscillating factor $\exp(i(2N+3)\omega)$ in the denominator the spectrum exhibits a typical line structure where the distance between adjacent lines is determined by the inverse mean length of laminar motions [4, 7]. The envelope of this spectrum is the quantity of interest in the subsequent discussion. It can be written as [7]

$$(I_q(\omega))_{env.} \simeq \mu_{N+1} \left(2\frac{1 - e^{q-q_c}\mathrm{Re}g(e^{i\omega}) + (1 - e^{q-q_c})|g(e^{i\omega})|}{1 - 2e^{q-q_c}\mathrm{Re}g(e^{i\omega}) - (1 - 2e^{q-q_c})|g(e^{i\omega})|^2} - 1 \right) \quad . \tag{11}$$

3 Envelope of the power spectrum

From the considerations presented in the preceeding section we can immediately determine the behaviour of the envelope in the asymptotic limit $N \to \infty$ as well as its dependence on the bifurcation parameter. The former can be done completely analytical in the limit of small frequencies whereas for the latter we refer back to numerical evaluations. The behaviour of the envelope for small frequencies is determined by the asymptotic expansion of the quantity (10) around $z = 1$. By a theorem of [15] this expansion depends only on the asymptotic behaviour $c_k \simeq Ck^{-a}$ that means on the order of the tangency. We obtain[3]

$$g(z) \simeq \begin{cases} 1-\alpha^{(\infty)}C\frac{\Gamma(2-a)}{a-1}(1-z)^{a-1} & 1<a<2, \quad 2<\zeta<\infty \\ 1+\alpha^{(\infty)}C\ln(1-z)-\alpha^{(\infty)}\beta(1-z) & a=2, \quad \zeta=2 \\ 1-g'(1)(1-z)+\alpha^{(\infty)}C\frac{\Gamma(3-a)}{(a-1)(a-2)}(1-z)^{a-1} & 2<a<3, \quad \frac{3}{2}<\zeta<2 \\ 1-g'(1)(1-z)-\alpha^{(\infty)}\frac{C}{2}(1-z)^2\ln(1-z) & a=3, \quad \zeta=\frac{3}{2} \\ 1-g'(1)(1-z)+\frac{1}{2}g''(1)(1-z)^2 & a>3, \quad 1\le\zeta<\frac{3}{2} \end{cases} \tag{12}$$

where $\Gamma(z)$ denotes the Gamma function. From this expansion the behaviour of the envelope for small frequencies can be immediately derived using eq.(11)

$$(I_q(\omega))_{env.} \simeq \begin{cases} \frac{1}{K_1(\zeta)}\omega^{-\frac{1}{\zeta-1}} & \frac{3}{2}<\zeta<\infty \\ -\frac{2}{C}\omega^{-2}(\ln\omega)^{-1} & \zeta=\frac{3}{2} \\ \frac{1}{K_2(q)}\omega^{-2} & 1<\zeta<\frac{3}{2} \end{cases} \tag{13}$$

where the coefficients are given by

$$\begin{aligned} K_1(\zeta) &:= \frac{\alpha^{(\infty)}C\pi}{2\Gamma(\frac{\zeta}{\zeta-1})\sin(\frac{\pi}{2(\zeta-1)})} \\ K_2(q) &:= \frac{1}{2}(g'(1)+g''(1))-\frac{1-2e^{q-q_c}}{2(1-e^{q-q_c})}(g'(1))^2 \end{aligned} \tag{14}$$

At the bifurcation point the spectrum obeys a power law at small frequencies with a q independent exponent. For a low order tangency $1<\zeta<3/2$ the laminar motion is rather short time correlated leading to a ω^{-2} decay as in the SOM map. Above the critical value $\zeta=3/2$ where logarithmic corrections to the power law occur the exponent decreases indicating a larger degree of periodicity among the laminar motion. No q dependence is however observed so that no local structure beyond the ordinary power spectrum can be detected on this level. This is not surprising as there are no significant unstable invariant sets among the laminar motion in our model of type I intermittency.

It is however important how this result is approached as N increases towards the bifurcation point. This behaviour can be derived from eqs.(6), (7) and (8) which

[3] $\beta := c_1+\sum_{k=2}^{\infty}k(c_k-C/k^2)$. In deriving the higher order expression (12_2) the asymptotic property $R(k) := k^2(c_k-C/k^2)$, $\lim_{k\to\infty}R(k)=0$ has been replaced by the slightly stronger condition $\sum^{\infty}|R(k)|/k<\infty$.

determine the N dependence of the eigenvalue λ_q, the maximum frequencies and the finite sum g_N. We have solved these equations for the parameter value $N = 200$ and have determined the envelope of the power spectrum for several values of q and ζ. First Fig. 2 shows the envelope for a fixed tangency $\zeta = 2$ and several q values. It < Fig.2
obviously obeys a power law for small frequencies. The corresponding exponents have been determined from a least square fit in the interval $\omega \in [0.04, 0.4]$ and are listed in table 1. They are nearly independent of q except for q values near the phase transition < Tbl.1
point $q_c = \ln 2$. The former is consistent with our results obtained at the bifurcation point whereas the latter comes from logarithmic corrections to the power law decay that emerge near the phase transition [7]. The dependence of the power spectra on the order of the tangency ζ is shown in Fig. 3. The corresponding exponents governing < Fig.3
the power law decay are depicted in Fig. 4 together with our analytical result eq.(13) < Fig.4
which is valid at the bifurcation point $N \to \infty$. Especially near the critical value $\zeta = 3/2$ a larger deviation is observed which comes from the logarithmic correction (13_2) to the power law decay. Furthermore the agreement becomes worse for very large ζ values as the detection of a high order tangency requires orbits with long laminar periods. As a consequence the convergency towards the analytical result is slower in both regions.

Due to a lack of an intrinsic structure in the laminar phase the power spectrum shows no significant q dependence. As can be guessed from our numerical estimate a weak q dependence might occur especially if experimental data are evaluated. On the one hand this dependence emerges due to an imperfect approach of the bifurcation point, on the other hand due to an insufficient resolution of the small frequency region. Both contributions lead to an error of the order ± 0.1 for the observed exponents. This fact should be kept in mind if datasets are analysed. The results can be improved if the dependence on the bifurcation parameter is taken into account by an appropriate scaling theory.

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References

[1] P.Manneville and Y.Pomeau, Physica D 1 (1980) 219

[2] C.Grebogi, E.Ott, F.Romeiras and J.A.Yorke, Phys. Rev. A 36 (1987) 5365

[3] H.G.Schuster, *Deterministic Chaos* (VCH, Weinheim, 1988)

[4] B.C.So and H.Mori, Physica D 21 (1986) 281

[5] Ya.G.Sinai, Uspekhi Math. Nauk 27 (1972) 27
R. Bowen, Lecture notes in mathematics, Vol. 470 *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, Berlin, 1975)
D.Ruelle, Encyclopedia of mathematics and its applications, Vol.5 *Thermodynamical Formalism* (Addison–Wesley, Reading, 1978)
P.Grassberger, R.Badii and A.Politi, J. Stat. Phys. 51 (1988) 109
H.Mori, H.Hata T.Horita and T.Kobayashi, Prog. Thoer. Phys. Suppl. 99 (1989) 1
M.J.Feigenbaum, I.Procaccia and T.Tel, Phys. Rev. A 39 (1989) 5359

[6] H.Fujisaka and H.Shibata, Prog. Theor. Phys. 85 (1991) 187

[7] W.Just and H.Fujisaka, Physica D (1992) *in press*

[8] N.Mori, T.Kobayashi, H.Hata, T.Morita and H.Mori, Prog. Theor. Phys. 81 (1989) 60

[9] K.Honda, S.Sato and H.Kodama, Phys. Rev. A 43 (1991) 2669

[10] K.Fukushima and T.Yamada, *preprint*

[11] P.Szepfalusy and T.Tel, Phys. Rev. A 34 (1986) 387
H.Fujisaka and M.Inoue, Prog. Theor. Phys. 78 (1987) 1334

[12] X.J.Wang, Phys. Rev. A 40 (1989) 6647

[13] H.Mori, B.C.So and T.Ose, Prog. Theor. Phys. 66 (1981) 1266

[14] T.Kobayashi, H.Fujisaka and W.Just, Phys. Rev. A *submitted*

[15] E.C.Titchmarsh, *The Theory of Functions* p.224f (Oxford Univ. Press 1939)

Figure captions:

Fig.1: Type I intermittent piecewise linear model map. $I_k^{(N)}$, $-N \leq k \leq N+1$ indicate the intervals of the Markov partition ($N = 3$). The length of the laminar motion N acts as the bifurcation parameter.

Fig.2: Envelope of the power spectrum for $\zeta = 2$ and several q values (from above $q = 0.5,\ 0.25,\ 0.0,\ -0.5,\ -1,\ -2,\ -4$). The rhombs indicate the poition of the maxima. For clearness adjacent spectra are shifted by the factor $10^{1/3}$.

Fig.3: Envelope of the power spectrum for $q = -2$ and several ζ values (from above $\zeta = 1.2,\ 1.4,\ 1.5,\ 1.6,\ 2,\ 3,\ 5$). The rhombs indicate the position of the maxima.

Fig.4: Exponents governing the power law decay for $N \to \infty$ (solid line) and $N = 200$, $q = -2$ (rhombs).

Table 1

q	-4	-2	-1	-0.5	0	0.25	0.5
exp.	1.00	0.95	0.95	0.97	1.04	1.11	1.18

Table 1: Exponents governing the algebraic decay of the power spectrum for $N = 200$ and $\zeta = 2$ for several q values (cf. Fig.2).