Analytical treatment of fluctuation spectra at the symmetry breaking chaos transition *

Wolfram Just
Theoretische Festkörperphysik
Technische Hochschule Darmstadt
Hochschulstraße 8
D– 6100 Darmstadt
Fed. Rep. Germany

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Abstract

The phase transition of characteristic exponents describing the fluctuations of temporal coarse grained quantities is investigated for the symmetry breaking bifurcation. The discussion of the eigenvalues of the transfer operator clarifies the origin of the phase transition and leads to scaling relations in the vicinity of the bifurcation point.

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1 Introduction

Symmetry breaking bifurcations of chaotic attractors have been observed in a variety of nonlinear systems [1]. For the investigation of this phenomenon characteristic exponents have been introduced [2], [3] to describe the stationary and transient fluctuations on the attractor. Even if the exponent λ_q governing the stationary fluctuations is not identical to the averaged local coarse grained expansion rate usually used in the thermodynamic formalism of dynamical systems [4], [5], [6] it shares a lot of properties with the latter. Especially phase transitions have been observed in the vicinity of the symmetry breaking bifurcation point and several scaling behaviours have been proposed [7]. In order to contribute to the problem whether there is a typical scaling behaviour in the vicinity of the band splitting point and what its functional form might be I will investigate simple one dimensional systems in this letter. The phase transition point and the scaling behaviour will be investigated in detail. Section 2 presents the model system, a piecewise linear tent like map. Furthermore the transfer operator technique is reviewed [3], [8] which yields the characteristic exponents by analytical computation. Section 3 contains the investigation of the spectrum of the transfer operator. This discussion clarifies the nature of the phase transition and leads to the scaling behaviour in the vicinity of the bifurcation point. In the $4^{th.}$ section the results are extended to the class of not necessarily piecewise linear expanding Markov maps. Finally the results are summarized and a generalization to certain nonhyperbolic situations is suggested.

2 Transfer operator method for a tent like map

To begin with let us consider the following simple piecewise linear one dimensional dynamical system (cf. Fig.1) $x_{n+1} = T_\epsilon(x_n)$

$$T_\epsilon(x) = \begin{cases} (2+\epsilon)x, & x \in [-a, a] \\ -(2+\epsilon)(x-2a), & x \in (a, 1] \\ -(2+\epsilon)(x+2a), & x \in [-1, -a) \end{cases} \quad , \qquad |T'_\epsilon(x)| = (2+\epsilon) =: \gamma \quad . \tag{1}$$

This system admits one chaotic attractor for $\epsilon > 0$ which splits into two different attractors in the limit $\epsilon \downarrow 0$. The main features this symmetry breaking bifurcation are contained in the fluctuations of the observable

$$u(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \quad . \tag{2}$$

For this reason the distribution of the coarse grained quantity $U_n(x) := \sum_{k=0}^{n-1} u(T_\epsilon^k(x))/n$

$$\rho_n(\alpha) = \langle \delta(U_n(x) - \alpha) \rangle \tag{3}$$

and the characteristic exponent

$$q\lambda_q := \lim_{n\to\infty} \frac{1}{n} \ln\langle \exp\left(qnU_n(x)\right)\rangle \tag{4}$$

have been introduced [2], [9]. In these equations the expectation value $\langle\cdots\rangle$ can be viewed as an averaging over a typical orbit of the map (1) or as an expectation value with respect to the Sinai– Ruelle– Bowen measure [4]. Using results of the large deviation theory the distribution (3) obeys the asymptotic relation [10]

$$\rho_n(\alpha) \sim \exp(-n\sigma(\alpha)) \quad . \tag{5}$$

The spectrum of fluctuations $\sigma(\alpha)$ and the characteristic exponent λ_q which governs the stationary correlations of the quantity U_n are related via [9]

$$\begin{aligned} q\lambda_q &= q\alpha(q) - \sigma(\alpha(q)) \\ \alpha(q) &= \frac{d}{dq}(q\lambda_q) \quad . \end{aligned} \tag{6}$$

It is the objective of the subsequent sections to investigate these quantities in the vicinity of the bifurcation point.

In order to keep the calculation analytical I will assume that the value of the bifurcation parameter ϵ of the map (1) allows for the existence of a Markov partition. This situation can be achieved if the extrema of the map and its iterates constitute an unstable periodic oribit. The parameter ϵ is restricted by the relation

$$T_\epsilon^m(a) = -a \quad \Rightarrow \quad 1 = \epsilon\gamma^{m-1} \qquad m = 2, 3, \ldots \quad . \tag{7}$$

The proposed sequence of parameter values behaves asymptotically as $\epsilon \simeq 2^{-m+1}$ and converges towards the bifurcation value. It is reasonable to assume that the typical behaviour of the bifurcation is contained in this sequence of ϵ values. For fixed ϵ that means for fixed m the map admits the following Markov partition (cf. Fig.1)

$$\begin{aligned} I_0^+ &= (0, c_0) \\ I_k^+ &= (c_0\gamma^{k-1}, c_0\gamma^k), \qquad 1 \le k \le m-2 \\ I_{m-1}^+ &= (a, \gamma a) \\ I_0^- &= (-c_0, 0) \\ I_k^- &= (-c_0\gamma^k, -c_0\gamma^{k-1}), \qquad 1 \le k \le m-2 \\ I_{m-1}^- &= (-\gamma a, -a) \end{aligned} \tag{8}$$

where $c_0 := \epsilon\gamma a$. The dynamics of this partition is governed by the equations

$$\begin{aligned} T_\epsilon(I_0^\pm) &= I_0^\pm \cup I_1^\pm \\ T_\epsilon(I_k^\pm) &= I_{k+1}^\pm, \qquad 1 \le k \le m-2 \\ T_\epsilon(I_{m-1}^\pm) &= \bigcup_{k=0}^{m-1} I_k^\pm \cup I_0^\mp \end{aligned} \tag{9}$$

The characteristic exponent can be computed with the help of the following transfer operator [3]

$$(\mathcal{H}_q^u f)(x) = \int \delta(T_\epsilon(y) - x) \exp(qu(y)) f(y) dy \tag{10}$$

The logarithm of the largest eigenvalue $\nu_q^{(0)}$ yields the desired quantity

$$\ln \nu_q^{(0)} = q\lambda_q \quad . \tag{11}$$

The remaining eigenvalues characterize the transient correlations of the coarse grained quantity U_n and describe the relaxation of an initial situation towards the stationary state via the damping rates

$$\gamma_q^{(l)} := \ln \nu_q^{(0)} - \ln \mathrm{Re}(\nu_q^{(l)}) \quad . \tag{12}$$

It is a hard task to diagonalize the operator (10) in general. But in the case considered here the characteristic functions of the intervals (8) build up a $2m$ dimensional invariant subspace of the operator $\mathcal{H}_q^u$ [8]. On this subspace the operator admits the $2m \times 2m$ dimensional matrix representation [1]

$$\gamma^{-1} \begin{pmatrix} \mathbf{A}(q) & \mathbf{B}(-q) \\ \mathbf{B}(q) & \mathbf{A}(-q) \end{pmatrix} \tag{13}$$

where

$$\mathbf{A}(q) := \exp(q) \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}_{m\times m} , \qquad \mathbf{B}(q) := \exp(q) \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}_{m\times m} \tag{14}$$

A short calculation yields the characteristic equation

$$(\mu_+^{(l)})^{m-1}(\mu_+^{(l)} - 2)(\mu_-^{(l)})^{m-1}(\mu_-^{(l)} - 2) = 1, \qquad \mu_\pm^{(l)} := \gamma e^{\mp q} \nu_q^{(l)} \tag{15}$$

where the index $l = 0, ..., 2m-1$ enumerates the different eigenvalues.

3 Spectrum of the transfer operator

With the help of the characteristic equation (15) the spectrum of the transfer operator can be investigated in the limit of asymptotically large m ($m \to \infty$). Let us first

[1] with respect to the canonical basis.

consider the case of fixed $q = 0$. As the second and fourth factor in eq.(15) cannot converge towards unity at the same time the two largest solutions of eq.(15) are obviously given by

$$\begin{aligned} 2 &\simeq \min\{\mu_+^{(0)}, \mu_-^{(0)}\} \\ 2 &\simeq \max\{\mu_+^{(1)}, \mu_-^{(1)}\} \end{aligned} \tag{16}$$

Furthermore one can argue that the remaining $2m-2$ solutions fulfil (appendix A)

$$|\gamma \nu_q^{(l)}| \simeq 1 \qquad l = 2, 3, ..., 2m-1 \quad . \tag{17}$$

Summarizing these foundings the eigenvalues of the transfer operator read [2] (cf. Fig.2)

$$\begin{aligned} \nu_q^{(0)} &\simeq \exp(|q|) \\ \nu_q^{(1)} &\simeq \exp(-|q|) \\ |\nu_q^{(l)}| &\simeq \frac{1}{2}, \qquad l = 2, 3, ..., 2m-1 \end{aligned} \tag{18}$$

and the characteristic exponent is easily obtained as

$$\lambda_q \simeq \begin{cases} 1, & q > 0 \\ -1, & q < 0 \end{cases} \quad . \tag{19}$$

It shows the well known step function behaviour. Inspecting the spectrum of the transfer operator (cf. eq.(18) or Fig.2) this phase transition is clearly understood by the crossing of two non degenerated eigenvalues at $q = 0$. To analyse this critical region in more detail the limits $m \to \infty$ and $q \to 0$ have to be considered simultaneously. Using eq.(7) the characteristc equation reads

$$\begin{aligned} &\exp\left[(m-1)(\ln \nu_q^{(l)} - q)\right] \left(\exp\left[\ln \nu_q^{(l)} - q\right] - \frac{2}{\gamma}\right) \\ \cdot \ &\exp\left[(m-1)(\ln \nu_q^{(l)} + q)\right] \left(\exp\left[\ln \nu_q^{(l)} + q)\right] - \frac{2}{\gamma}\right) = \frac{\epsilon^2}{\gamma^2} \end{aligned} \tag{20}$$

As we are interested in the critical region $q \approx 0$ and in the relevant eigenvalues $\nu_q^{(0/1)} \approx 1$ the following scaling is appropriate

$$q =: \epsilon \frac{\kappa}{2}, \qquad \ln \nu_q^{(i)} =: \epsilon \frac{\psi_q^{(i)}}{2}, \quad i = 0, 1 \tag{21}$$

[2] One has to note that this result is valid on the subspace described in section 2. There is no indication that the result changes qualitatively if one takes the whole space into account.

where κ and $\psi_q^{(i)}$ are of the order $O(1)$ in the limit $\epsilon \downarrow 0$. Then eq.(20) yields

$$(\psi_q^{(i)} - \kappa + 1)(\psi_q^{(i)} + \kappa + 1) = 1 + O(\epsilon \ln \epsilon) \tag{22}$$

From this equation the scaling relations of the two relevant eigenvalues which govern the long time dynamics in the vicinity of the bifurcation point are immediately obtained as

$$\begin{aligned} \ln \nu_q^{(0)} = \frac{\epsilon}{2}\psi_q^{(0)} &\simeq \frac{\epsilon}{2}(\sqrt{1+\kappa^2} - 1) \\ \ln \nu_q^{(1)} = \frac{\epsilon}{2}\psi_q^{(1)} &\simeq \frac{\epsilon}{2}(-\sqrt{1+\kappa^2} - 1), \quad \kappa = \frac{2q}{\epsilon} \end{aligned} \tag{23}$$

Due to the fact that one has considered asymptotically small but non vanishing ϵ values the degeneracy of the eigenvalues at the transition point $q = \kappa = 0$ has been removed. Furthermore the scaling relations for the characteristic exponent λ_q, the spectrum of the fluctuations and the damping rate follow from eqs.(6), (11), (12) and (23)

$$\begin{aligned} \lambda_q &\simeq \frac{\sqrt{1+\kappa^2} - 1}{\kappa} \\ \sigma(\alpha) &\simeq \frac{\epsilon}{2}(1 - \sqrt{1-\alpha^2}) \\ \gamma_q^{(1)} &\simeq \frac{\epsilon}{2} 2\sqrt{1+\kappa^2}, \quad \kappa = \frac{2q}{\epsilon} \end{aligned} \tag{24}$$

The result (24) has been obtained with the help of a phenomenological stochastic two state model for general systems [11]. This is of course not surprising since our deterministic model system (1) shares a lot of properties with a stochastic two state system.

4 Generalization to expanding Markov maps

The scaling result $(24_{1,2})$ obtained in the preceeding section for the special model system (1) can be extended to the rather large class of inversion symmetrical expanding Markov maps. As will be shown in this section these maps are conjugate to the example treated in section 2 and admit the same characteristic exponent λ_q. The construction of the homeomorphism will be achieved by tracing back to the symbolic dynamics induced by the Markov partition.

Let $\tilde{T}_\epsilon$ be an inversion symmetrical expanding Markov map ($|\tilde{T}'_\epsilon| \geq c > 1$) on the interval [3] $[-1, 1]$ which admits a Markov partition $\tilde{I}_k^{\pm}, 0 \leq k \leq m-1$. Its dynamics

[3] This assumption is only technical and can be removed.

should be governed by equations analogous to eq.(9). Fig.3 shows a sketch of such a map. Consider the dynamic partition

$$\tilde{U}_{i_0\sigma_0,\ldots,i_{n-1}\sigma_{n-1}} = \{\tilde{x} \in [-1,1] \,|\, \tilde{T}_\epsilon^k(\tilde{x}) \in \tilde{I}_{i_k}^{\sigma_k}, 0 \le k \le n-1\} \tag{25}$$

where $i_k \in \{0,...,m-1\}, \sigma_k = \pm$ and $i_0\sigma_0,...,i_{n-1}\sigma_{n-1}$ denote the symbol sequences which are allowed by the transition matrix of the Markov partition (cf. eq.(13) in the case $q = 0$ or eq.(9)). Take a symbol sequence and regard the limit $n \to \infty$. Then the sequence of intervals (25) single out one point $x \in [-1,1]$ because their lengths decrease exponetially by the expansiveness of the map $\tilde{T}_\epsilon$. By this construction every allowed symbol sequence $i_0\sigma_0, i_1\sigma_1, \ldots$ is maped onto one point $x \in [-1,1]$. This prescription is invertible if the countable set of points is neglected that build up the stable manifold of the endpoints of the intervals $\tilde{I}_k^\pm, 0 \le k \le m-1$. The bijection constituted in this way is continuous with respect to the canonical metric in the space of symbol sequences (cf. appendix B).

Let us consider now two maps $\tilde{T}_\epsilon$ and T_ϵ possessing a Markov partition with the same transition matrix. If one applies the construction described above to both one gets by composing the two bijections a continuous invertible function $h : [-1,1] \to [-1,1]$ which conjugates the two maps.

$$h \circ T_\epsilon = \tilde{T}_\epsilon \circ h \tag{26}$$

It is the main point that the characteristic exponent (4) is invariant with respect to the continuous (not necessarily Lipschitz continuous or differentiable) transformation h. If $\tilde{\lambda}_q$ denotes the characteristic exponent of $\tilde{T}_\epsilon$ then eqs.(4) and (26) yield

$$q\tilde{\lambda}_q = \lim_{n\to\infty} \frac{1}{n} \ln\langle\exp\left(q\sum_{k=0}^{n-1} u(h(T_\epsilon^k(x)))\right)\rangle \tag{27}$$

In deriving eq.(27) one difficulty arises via the meaning of the expectation values. This rather technical point is devoted to appendix B. As h leaves invariant the intervals $[-1,0]$ and $[0,1]$ (cf. appendix B) the relation $u = u \circ h$ holds. Therefore the characteristic exponent is invariant under the transformation h and given by eq.(24_1) in the vicinity of the symmetry breaking bifurcation point. If this result is influenced by any details of the dynamical system it could depend only on the form of the transition matrix. Finally it should be stressed that the invariance of the characteristic exponent λ_q relies on a continuous bijection. This is in contrast to other kinds of characteristic exponents used in the literature [6], [12] where Lipschitz continuity is required for the invariance under a transformation.

5 Short summary and remarks

In this letter the phase transition in the vicinity of the symmetry breaking bifurcation point has been discussed for one dimensional maps with analytical methods. It was shown that the phase transition is accomplished by a degeneracy of two simple eigenvalues of the transfer operator. This situation which I believe is typical for the bifurcation allows for a description by two relevant eigenvalues and is the origin of the scaling behaviour (24) of several characteristic exponents. The validity of the scaling relations ($24_{1,2}$) has been extended strictly to the class of expanding Markov maps. Tracing back to numerical calculations on nonhyperbolic systems and differential equations it should be emphasized that the scaling relations can be generalized to nonhyperbolic situations at least for special cases. Consider for example a one dimensional map with smooth extrema in the vicinity of a symmetry breaking bifurcation. If the extrema fall into the stable manifold of an unstable orbit so that an absolutely continuous invariant measure exists [13] the considerations of section 4 can be applied to this case also. But these situations together with the discussion of higher dimensional systems demands for further investigations.

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Appendix A

Consider the characteristic equation (15)

$$(\mu^{(l)})^{2m-2}(\exp(q)\mu^{(l)}-2)(\exp(-q)\mu^{(l)}-2)=1, \quad \mu^{(l)}:=\gamma\nu_q^{(l)} \quad . \tag{28}$$

We will show that there exist $2m-2$ solutions with $|\mu^{(l)}|\simeq 1$. For this reason introduce $\mu^{(l)}=r^{(l)}\exp(i\delta^{(l)})$ with the assumption $r^{(l)}\to 1$ and $(r^{(l)})^{m-1}\to\alpha^{(l)}$ in the limit $m\to\infty$. Then eq.(28) reads in the asymptotic limit of large m

$$\alpha^{(l)}\exp(i(2m-2)\delta^{(l)})\left(\exp(q)\exp(i\delta^{(l)})-2\right)\left(\exp(-q)\exp(i\delta^{(l)}-2\right)\simeq 1 \quad . \tag{29}$$

That means

$$\exp(i(2m-2)\delta^{(l)}) \simeq \frac{\left(2-\exp(q)\exp(-i\delta^{(l)})\right)\left(2-\exp(-q)\exp(-i\delta^{(l)})\right)}{|\,(2-\exp(q)\exp(-i\delta^{(l)}))\,(2-\exp(-q)\exp(-i\delta^{(l)}))\,|} \quad . \tag{30}$$

To show that this equation admits $2m-2$ solutions in the vicinity of the phase transition point $|q| < \mathrm{Arcosh}\, 5$ [4] divide the unit circle $[0, 2\pi)$ into $2m-2$ arcs $D_l = (\pi(2l-5)/(2m-2), \pi(2l-3)/(2m-2))$, $l = 2, ..., 2m-1$ and consider a fixed arc. On this arc the argument of the left hand side of eq.(30) varies continuously from $-\pi$ to π whereas the argument of the right hand side varies in the interval $(-\pi/2, \pi/2)$. As a consequence there exists for every l some $\delta^{(l)} \in D_l$ which solves eq.(30) and leads to the solutions $\mu^{(l)}$ proposed above.

Appendix B

Let $\tilde{\pi}$ denote the map from the space of allowed symbol sequences equiped with the canonical metric

$$\begin{aligned} d(i_0\sigma_0, ...|j_0\tau_0, ...) &:= \sum_{k=0}^{\infty} 2^{-k}(|i_k - j_k| + d(\sigma_k, \tau_k)) \\ d(\sigma_k, \tau_k) &:= \left\{ \begin{array}{ll} 1, & \text{if}\, \sigma_k = \tau_k \\ 0, & \text{if}\, \sigma_k = \tau_k \end{array} \right. \end{aligned} \tag{31}$$

onto the interval $[-1, 1]$. Consider two symbol sequences with a distance smaller than δ. Then they coincide in the first N_δ symbols where N_δ can be made sufficiently large by chosing δ small. Hence their images $\tilde{x} = \tilde{\pi}(i_0\sigma_0, ...)$ and $\tilde{y} = \tilde{\pi}(j_0\tau_0, ...)$ lie in the same interval (25) of the dynamical partition (with $n = N_\delta$) and their distance can be made arbitrary small. This proves the continuity of $\tilde{\pi}$. The continuity of the inverse map can be shown in a similar way.

Consider the transformations π and $\tilde{\pi}$ of the two maps mentioned in section 4. Then $h := \tilde{\pi} \circ \pi^{-1}$ yields the homeomorphism between the maps. Let $x \in [0, 1]$ be chosen arbitrarily. Then by definition $\pi^{-1}(x) = i_0+, i_1\sigma_1, ...$ and $\tilde{\pi}(\pi^{-1}(x)) \in [0, 1]$. Hence the intervals $[0, 1]$ and $[-1, 0]$ are invariant with respect to h.

To derive eq.(27) one starts from the definition of $\tilde{\lambda}_q$ (cf. eq.(4)). Here the expectation value has to be considered with respect to the SRB measure of $\tilde{T}_\epsilon$ or as an averaging over a trajectory for (Lebesque-) almost all initial points $\tilde{x}$. We adopt

[4]This constraint follows from the requirement that the real part of the right hand side in eq.(30) is always positive.

the latter formulation for simplicity and get with the help of eq.(26)

$$q\tilde{\lambda}_q = \lim_{n\to\infty} \frac{1}{n} \ln \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \exp\left(q \sum_{k=0}^{n-1} u(h(T_\epsilon^k(T_\epsilon^i(x)))\right) \quad . \tag{32}$$

By the continuity of h this relation is valid for (Lebesque-) almost all initial points x. Using the property of the SRB measure that the average over the trajectory of T_ϵ can be written as an phase space average one gets eq.(27).

Figure captions

Fig.1 Piecewise linear map (1). A Markov partition for the case $m = 3$ is indicated. $a < 1/2$, $0 \leq \epsilon \ll 1$.

Fig.2 Diagrammatic view of the spectrum of the transfer operator $\mathcal{H}_q^u$. ( ) largest eigenvalue, ( ) second eigenvalue, ( ) remaining eigenvalues (absolute value).

Fig.3 Expanding Markov map in the vicinity of a symmetry breaking bifurcation.

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