Gibbs Measures and Power Spectra for Type I Intermittent Maps

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Abstract

Gibbs measures of dynamical systems are investigated using the corresponding correlation function and power spectrum. The method is shown to be equivalent to the concept of order–q power spectra which has been developed previously. A technique for the computation of these spectra is presented. As an application we specially focus on the treatment of type I intermittent model maps and investigate their phase transition. In the vicinity of the phase transition point the order–q power spectra are shown to obey some typical scaling relation which is captured neither by the topological pressure nor by the ordinary power spectrum.

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1 Introduction

Two decades ago the thermodynamical formalism has been developed in the context of the mathematical formulation of equilibrium statistical mechanics and the ergodic theory of dynamical systems [1, 2, 3]. Recently this method has become popular among physicists in analysing nonlinear dynamical systems [4, 5]. One of the main aims of this method consists in investigating the different invariant measures of a certain dynamical system to get information about the local structure of the chaotic invariant set. For this purpose the variational principle [6]

$$P_T(\varphi) = \sup \left\{ h_\mu(T) + \int \varphi d\mu \right\} \tag{1}$$

has been used, where T denotes a discrete dynamical system $x_{n+1} = T(x_n)$ [1] on some metric space, φ a (piecewise) continuous function, $h_\mu(T)$ the Kolmogorov Sinai entropy and the supremum is taken with respect to all T invariant measures μ. Under certain mathematical conditions the supremum is attained at a measure μ_φ called Gibbs measure, which depends on the chosen function φ. The topological pressure $P_T(\varphi)$ may admit different analytical branches corresponding to different local structures of the invariant set [7, 8]. These structures are reflected by the different Gibbs measures. To investigate the quantity (1) a transfer operator $\mathcal{L}_\varphi$ and its formally adjoint $\mathcal{L}_\varphi^\dagger$ acting on the space of continuous functions respectively on the space of measures have been introduced [3]. Its eigenvalue problem

$$\mathcal{L}_\varphi h_\varphi^{(0)} = \lambda_\varphi^{(0)} h_\varphi^{(0)} \qquad \mathcal{L}_\varphi^\dagger \nu_\varphi^{(0)} = \lambda_\varphi^{(0)} \nu_\varphi^{(0)} \tag{2}$$

for the largest necessarily positive eigenvalue yields the topological pressure and the Gibbs measure via the relations

$$P_T(\varphi) = \ln \lambda_\varphi^{(0)}, \qquad d\mu_\varphi = h_\varphi^{(0)} d\nu_\varphi^{(0)} \quad . \tag{3}$$

But explicit expressions of this transfer operator have been given only for special cases like one dimensional or expanding maps or shift systems of the finite type [9].

A different but widely equivalent approach often used by physicists is based on the investigation of the large fluctuations of temporal coarse grained quantities $U_n(x) = \sum_{i=0}^{n-1} u(T^i(x))/n$. Here T^i denotes the i times iterated map and $u(x)$ the quantity under consideration. The large fluctuations are appropriately described by the characteristic function [10, 4]

$$\Phi(q) = \lim_{n\to\infty} \frac{1}{n} \ln\langle \exp(qnU_n(x))\rangle \tag{4}$$

and its derivatives $\langle u\rangle(q) := \Phi'(q)$, $\chi(q) := \Phi''(q)$. The average $\langle \ldots \rangle$ over the initial points is usually taken as a long time average or an average with respect to the SRB measure. For a large class of systems (axiom A, one dimensional expanding maps) the quantities (1) and (4) have proven to be identical if one identifies $\varphi = qu - \Lambda$ [2], where Λ denotes the logarithm of the local expansion rate [11]. Nevertheless it should be pointed out that even in one

[1] A generalization to time continuous systems is straightforward.

[2] In the sequel we use $\varphi := qu - \Lambda$ as an abbreviation.

dimensional non–hyperbolic maps both quantities may differ, reflecting a dependence of the quantity (4) on the measure which constitutes the average $\langle \ldots \rangle$ [4, 8, 12]. In general the characteristic function can be related to the eigenvalue problem of the transfer operator [13]

$$(\mathcal{H}_q^u h)(x) = \int \delta(x - T(y)) \exp(qu(y)) h(y) dy \tag{5}$$

acting on the space of (piecewise) continuous functions. The largest eigenvalue yields the quantity (4) in the same way as described by eq.(3). It should be stressed that the identity of the operators $\mathcal{L}_\varphi$ and $\mathcal{H}_q^u$ have been shown only for one dimensional systems. To our knowledge it is an open question whether there exists any relation for higher dimensional maps.

The considerations presented above demonstrate that the quantities (1) and (4) contain only the stationary information concerning the Gibbs measure. In order to deal with the time correlations of these dynamical systems which are beyond the usual double time correlation approach [14] one of the authors has introduced a weighted correlation function and the corresponding power spectrum [15]

$$\begin{aligned} C_q(j) &= \lim_{k\to\infty} \lim_{n\to\infty} \langle u(T^{j+k}(x)) u(T^k(x)) \exp(qnU_n(x))\rangle / \langle \exp(qnU_n(x))\rangle \\ I_q(\omega) &= \lim_{n\to\infty} \langle \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} u(T^k(x)) e^{-i\omega k} \right|^2 \exp(qnU_n(x))\rangle / \langle \exp(qnU_n(x))\rangle \quad . \end{aligned} \tag{6}$$

It has been shown that these quantities, called order–q correlation function and –power spectrum, are determined by all the eigenvalues of the operator (5). For this reason it is highly probable that they contain information about the dynamical system which is not contained in the description given by the topological pressure.

It is one main aim of this article to show how the quantities (6) are related to the Gibbs measures presented at the beginning. This goal will be achieved in section 2 where together with a new interpretation of the quantities (6) an efficient method for their computation via the transfer operator (5) will be presented. In order to avoid the several mathematical pitfalls which have been mentioned in this introduction we will concentrate on the treatment of one dimensional maps. Nevertheless we want to stress that most of our considerations can be carried over to higher dimensional systems of the axiom A type without great effort. In the remainder of this article we will focus on type I intermittent maps which are known to have a remarkable structure in their ordinary power spectrum [16] and are well studied concerning their meromorphic structure [17]. In order to keep the calcualtions essentially analytical we construct a piecewise linear model for this non–hyperbolic case in section 3. Furthermore we give a detailed discussion of the phase transition emerging near the bifurcation point from the viewpoint of the corresponding Gibbs measures. The power spectra in the different phases will be analysed in section 4. The investigation of the immediate vicinity of the phase transition point is devoted to section 5. We will finish our article with some concluding remarks.

2 Generalized power spectrum

At the beginning we want to state the notation. Let T denote a one dimensional map, u a piecewise continuous function and ν a not necessarily T invariant measure. By $\langle\nu|h\rangle$ we denote the bilinear expression $\int h d\nu$ acting on the function space and its dual. For an arbitrary T invariant measure μ let us consider the double time correlation function

$$C(k;\mu) := \langle\mu|u\cdot(u\circ T^k)\rangle = \int u(T^k(x))u(x)d\mu \tag{7}$$

and the corresponding power spectrum [3]

$$I(\omega;\mu) := \lim_{n\to\infty}\langle\mu|\frac{1}{n}\left|\sum_{k=0}^{n-1} e^{-i\omega k}\cdot(u\circ T^k)\right|^2\rangle \tag{8}$$

which are as usual related via the Wiener–Khinchin theorem

$$\begin{aligned} C(|k|;\mu) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} I(\omega;\mu)e^{-i\omega k}d\omega \\ I(\omega;\mu) &= \sum_{k=-\infty}^{\infty} C(|k|;\mu)e^{i\omega k} \quad . \end{aligned} \tag{9}$$

The meaning of these quantities is clear from their definition. They characterize the double time correlations of the transformation T with respect to the invariant measure μ which may not be the natural one [18]. But in the case of the SRB measure these quantities coincide with the ordinary correlation function and power spectrum. Let us further mention that the spectrum (9_2) contains a δ contribution at $\omega = 0$ as long as the relation $\langle\mu|u\rangle = 0$ does not hold.

In order to relate these objects to the quantities (6) two main steps have to be performed. Although the limits in the definition (6_1) may not be interchanged it can be shown that the order–q correlation function can be obtained by the following Cesaro like limit (cf. [19] for similar results)

$$C_q(j) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\langle u(T^{j+k}(x))u(T^k(x))\exp(qnU_n(x))\rangle/\langle\exp(qnU_n(x))\rangle \quad . \tag{10}$$

Furthermore the following relation between the q–weighted average of an arbitrary function g and the average with respect to the Gibbs measure is valid [2, 6, p.219]

$$\lim_{n\to\infty}\langle\frac{1}{n}\sum_{k=0}^{n-1} g(T^k(x))\exp(qnU_n(x))\rangle/\langle\exp(qnU_n(x))\rangle = \int g d\tilde{\mu}_q \quad . \tag{11}$$

Here $\tilde{\mu}_q$ denotes the Gibbs measure with respect to the function $\varphi = qu - \ln|T'|$. Both relations can be obtained also in an elementary way by using the spectral decomposition of the correlation function which is given in [15]. Using eqs.(6_1), (7), (10) and (11) we obtain

$$C_q(k) = C(k;\tilde{\mu}_q) \quad . \tag{12}$$

[3] $\omega \in (-\pi,\pi)$, $I(\omega;\mu) = I(-\omega;\mu)$.

From the validity of the Wiener–Khinchin theorem for the quantities (6) (cf. ref.[15]) we further conclude

$$I_q(\omega) = I(\omega; \tilde{\mu}_q) \quad . \tag{13}$$

As a consequence of this relation the order–q power spectrum $I_q(\omega)$ reflects directly the time correlations associated with the corresponding Gibbs measure. Furthermore the relations (12) and (13) allow for a computation of these quantities by the transfer operator $\mathcal{L}_\varphi = \mathcal{H}_q^u$. For this purpose we first mention that in view of the definition (5) for arbitrary functions f and g the relation

$$g(x)(\mathcal{H}_q^u f)(x) = \int g(T(y))\delta(x - T(y))e^{qu(y)} f(y)dy = (\mathcal{H}_q^u((g \circ T) \cdot f))(x) \tag{14}$$

holds, which by induction results in

$$g(x)((\mathcal{H}_q^u)^k f)(x) = (\mathcal{H}_q^u)^k((g \circ T^k) \cdot f)(x) \quad . \tag{15}$$

Referring back to the representation (3) of the Gibbs measure by eigenelements of the transfer operator we obtain for the correlation function (12)

$$\begin{aligned} C_q(k) &= \langle \nu^{(0)} | (u \circ T^k) \cdot u \cdot h^{(0)} \rangle = \langle \nu^{(0)} | (\mathcal{H}_q^u / \lambda_q^{(0)})^k ((u \circ T^k) \cdot u \cdot h^{(0)}) \rangle \\ &= \langle \nu^{(0)} | u \cdot (\mathcal{H}_q^u / \lambda_q^{(0)})^k (u \cdot h^{(0)}) \rangle = \int u(x) \left(\frac{\mathcal{H}_q^u}{\lambda_q^{(0)}} \right)^k (u \cdot h^{(0)})(x) d\nu^{(0)} \quad . \end{aligned} \tag{16}$$

Using the Wiener–Khinchin theorem (9) the order–q power spectrum reads

$$\begin{aligned} I_q(\omega) &= \langle \nu^{(0)} | u \cdot \left\{ (1 - \frac{e^{i\omega}}{\lambda_q^{(0)}} \mathcal{H}_q^u)^{-1} + (1 - \frac{e^{-i\omega}}{\lambda_q^{(0)}} \mathcal{H}_q^u)^{-1} - 1 \right\} (u \cdot h^{(0)}) \rangle \\ &= \int u(x) \left\{ (1 - \frac{e^{i\omega}}{\lambda_q^{(0)}} \mathcal{H}_q^u)^{-1} + (1 - \frac{e^{-i\omega}}{\lambda_q^{(0)}} \mathcal{H}_q^u)^{-1} - 1 \right\} (u \cdot h^{(0)})(x) d\nu^{(0)} \quad . \end{aligned} \tag{17}$$

The computation of the power spectrum has now been reduced to the evaluation of the resolvent of the transfer operator. At least in special examples this seems to be easier to evaluate than the computation of the whole spectrum which is desired to apply the method proposed in [15]. Finally we stress the well known fact that in the case of piecewise linear Markov maps the operator (5) admits a finite dimensional matrix representation [20]. In this case eq.(17) reduces to a matrix equation. The details of this case are briefly outlined in appendix A.

3 Type I intermittency and phase transition

Intermittency, which is a very common phenomenon from the experimental point of view, can be analysed in theoretical terms by low dimensional systems. Type I intermittency is known for a long time to be related to a saddle node bifurcation and can therefore be described by one dimensional maps [21]. We want to investigate this phenomenon in our framework presented above. In order to keep our calculations basically analytical we will construct a piecewise linear model map (cf. [22]).

As the lifetime of laminar states tends to infinity the map producing intermittency passes unstable periodic orbits of increasing length $a_k^{(N)}$, which can be used to construct a Markov partition according to the intervals $I_k^{(N)} = (a_{k-1}^{(N)}, a_k^{(N)})$. On these intervals the map is assumed to be linear a presupposition which is arbitrarily well satisfied in the vicinity of the bifurcation point (cf. Fig.1). If $\Delta a_k^{(N)} = a_k^{(N)} - a_{k-1}^{(N)}$ denotes the length of these intervals > Fig.1
then the slope of the map on these sets is given by $\gamma_k^{(N)} = \Delta a_{k+1}^{(N)} / \Delta a_k^{(N)}$, $(\Delta a_{N+2}^{(N)} := 2)$. We have imposed some symmetry constraints on the map $a_k^{(N)} = -a_{-k}^{(N)}$ in order to simplify the considerations. Our results are not essentially influenced by this requirement. As the laminar lifetime $2N$, which acts as a bifurcation parameter in our map, increases the values $a_k^{(N)}$ may change even for fixed k. To achieve the N dependence in a uniform way we require that the interval widths are given by

$$\Delta a_{N+1-k}^{(N)} = \Delta a_{-N+k}^{(N)} = c_k \alpha^{(N)}, \quad 0 \leq k \leq N \tag{18}$$

with a N independent sequence c_k. The normalization constant $\alpha^{(N)}$ is determined by

$$1 = \sum_{k=1}^{N+1} \Delta a_k^{(N)} = \alpha^{(N)} \sum_{k=0}^{N} c_k \quad . \tag{19}$$

The asymptotic behaviour of the sequence c_k for large k determines the order of the tangency of the map in the limit of large N. If $c_k \simeq k^{-a}$, $a > 1$ then it can be checked easily that the order of the tangency is given by $a/(a-1)$. As we are interested in quadratic tangencies we will require $c_k \simeq k^{-2}$. To be specific we will choose in our derivations and numerical calculations $c_k = 1/[(k+1)(k+2)]$ although our analytical results are independent of this special choice. To complete our model we have to specify the function u. Of course we can deal with a piecewise constant function taking arbitrary values on the intervals of the Markov partition. But we will concentrate for simplicity from the beginning on the simple choice of a function taking values 0 and 1 in the laminar respectively turbulent regions,

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in \bigcup_{k=-N}^{N} I_k^{(N)} \\ 1 & \text{if} \quad x \in I_{N+1}^{(N)} \end{cases} \quad . \tag{20}$$

Main parts of our results do not depend on this choice.

With these settings we can write down the finite matrix representation of the transfer operator (cf. appendix A)

$$\underline{\underline{H}}_q^u = \begin{pmatrix} 0 & & \cdots & 0 & e^q/\gamma_{N+1}^{(N)} \\ 1/\gamma_{-N}^{(N)} & 0 & & & \\ 0 & 1/\gamma_{-N+1}^{(N)} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & 1/\gamma_N^{(N)} & e^q/\gamma_{N+1}^{(N)} \end{pmatrix} \quad . \tag{21}$$

Its characteristic equation determining the eigenvalues λ_q can be obtained after a straightforward computation,

$$1 = \alpha^{(N)} \frac{e^q}{2} \left\{ \sum_{k=0}^{N} c_k \lambda_q^{-(2N+2-k)} + \sum_{k=0}^{N} c_k \lambda_q^{-(k+1)} \right\} \quad . \tag{22}$$

Furthermore the right and left eigenvectors h_k and ν_k which are related to the eigenfunctions and eigenmeasures of the transfer operator and its adjoint via the relations $h(x) = \sum_{k=-N}^{N+1} h_k \chi_k(x)$ [4] and $\nu_k = \nu(I_k^{(N)})$ are easily written down as

$$\begin{aligned} h_k &= \frac{\Delta a_{N+1}^{(N)}}{\Delta a_k^{(N)}} \lambda_q^{N+1-k} \frac{e^q}{2} \sum_{j=-N}^{k} \Delta a_j^{(N)} \lambda_q^{-(N+2-j)} h_{N+1} \\ \nu_k &= \frac{\Delta a_k^{(N)}}{\Delta a_{N+1}^{(N)}} \lambda_q^{-(N+1-k)} \nu_{N+1} \quad . \end{aligned} \tag{23}$$

From these equations we get the following result for the Gibbs measure $\mu_k := \tilde{\mu}_q(I_k^{(N)}) = h_k \nu_k$ of the interval $I_k^{(N)}$

$$\begin{aligned} \mu_{-N+k} &= \alpha^{(N)} \frac{e^q}{2} \sum_{j=0}^{k} c_j (\lambda_q^{(0)})^{-(2N+2-j)} \mu_{N+1} \\ \mu_{N+1-k} &= \left\{ 1 - \alpha^{(N)} \frac{e^q}{2} \sum_{j=0}^{k-1} c_j (\lambda_q^{(0)})^{-(j+1)} \right\} \mu_{N+1}, \qquad 0 \le k \le N \quad . \end{aligned} \tag{24}$$

The expression has been written in two different parts for simlicity and $\lambda_q^{(0)}$ denotes the largest solution of eq.(22) which is positive by Perron's theorem. μ_{N+1} is determined by the normalization condition

$$1 = \sum_{k=-N}^{N+1} \mu_k = \alpha^{(N)} \frac{e^q}{2} \left\{ \sum_{k=0}^{N} (2N+2-k) c_k (\lambda_q^{(0)})^{-(2N+2-k)} + \sum_{k=0}^{N} (k+1) c_k (\lambda_q^{(0)})^{-(k+1)} \right\} \mu_{N+1} \quad . \tag{25}$$

To get the explicit expressions for the topological pressure (4) and the Gibbs measure (24) we have to calculate the largest solution of eq.(22). As the coefficients of this polynomial are positive and depend on q in a monotonous way this solution is an increasing function of q. Especially the asymptotic limit $N \to \infty$ on which we will concentrate in this section is of special interest because a phase transition in the vicinity of the bifurcation point is expected, which reflects the different local structures of the attractor [20, 22, 23]. First of all $\lambda_q^{(0)} \ge 1$ holds in this limit. If $\lambda_q^{(0)} > 1$ for some q value above a critical value q_c then eq.(22) yields

$$1 = \frac{e^q}{2} g\left(\frac{1}{\lambda_q^{(0)}}\right) \tag{26}$$

[4] χ_k denotes the characteristic function of the interval $I_k^{(N)}$.

with the abbreviation [5]

$$g(z) = \alpha^{(\infty)} \sum_{k=0}^{\infty} c_k z^{k+1} = 1 + \frac{1-z}{z} \ln(1-z) \quad . \tag{27}$$

Here q_c is determined by the requirement that $\lambda_q^{(0)}$ approaches 1 from above. In view of eqs.(19) and (26) this leads to

$$q_c = \ln 2 \quad . \tag{28}$$

Inspecting this result one recognizes that at $q = q_c$ a non–analyticity that means a phase transition occurs which separates two different branches (phases) of the analytical behaviour. Below the phase transition point $\lambda_q^{(0)}$ equals unity whereas $\lambda_q^{(0)} > 1$ in the case $q > q_c$. The situation slightly above the phase transition point can be easily read off from eq.(26). For $0 \leq q - q_c, \lambda_q^{(0)} - 1 \ll 1$ we obtain the asymptotic relation [6]

$$\lambda_q^{(0)} - 1 \simeq -\frac{q - q_c}{\ln(q - q_c)}, \qquad \Phi(q) = \ln \lambda_q^{(0)} \simeq -\frac{q - q_c}{\ln(q - q_c)} \tag{29}$$

which depends only on the order of the tangency but not on any details of the map that means on details of the sequence c_k. As our function u possesses vanishing derivative at the tangency point the result (29) differs slightly from the behaviour that will emerge if the local expansion rate will be chosen as the observable [23]. But in both cases the phase transition is indicated by a divergency occuring in the second derivative $\chi(q)$ of the characteristic function. For completing our discussion we have solved eq.(22) numerically for several values of the bifurcation parameter N. Fig.2 shows the typical behaviour of the characteristic function and its derivatives near the phase transition point. As N increases the quantities approach the $N \to \infty$ limit discussed above. > Fig.2

To get more insight into the nature of the phase transition let us investigate the Gibbs measures (24) in more detail. In the case $q > q_c$ the relation $\lambda_q^{(0)} > 1$ holds for large N so that the normalization condition (25) reduces to

$$\begin{aligned} 1 &= \alpha^{(\infty)} e^{q-q_c} \sum_{k=0}^{\infty} (k+1) c_k (\lambda_q^{(0)})^{-(k+1)} \mu_\infty = e^{q-q_c} \frac{1}{\lambda_q^{(0)}} g'(\frac{1}{\lambda_q^{(0)}}) \mu_\infty \\ &= e^{q-q_c} \{-1 - \lambda_q^{(0)} \ln(1 - \frac{1}{\lambda_q^{(0)}})\} \mu_\infty \end{aligned} \tag{30}$$

which yields a finite value for $\mu_\infty := \lim_{N\to\infty} \mu_{N+1}$. Furthermore according to eq.(24) μ_k tends to zero for $k \leq 0$ but remains finite for positive k in the limit $N \to \infty$. This means that the measure is concentrated in the region $x > 0$ and is supported by the repelling invariant set (cf. eq.(48)). For this reason this phase is called the turbulent or hyperbolic phase. In the opposite case $q < q_c$ we have $\lambda_q^{(0)} \to 1$ in the limit $N \to \infty$. Then the normalization condition (25) yields $\mu_{N+1} = O(N^{-1})$ (for the detailed computation we refer to appendix

[5] In defining the logarithm the complex plane is slashed along the negative real axis.

[6] Using a result of [24] it can be shown that this relation depends only on the asymptotic behaviour $c_k \simeq k^{-2}$.

B). Furthermore eq.(24) leads to $\mu_k = O(N^{-1})$ because the sums remain finite in view of the eigenvalue equation. As only a finite number of the intervals $I_k^{(N)}$ of the Markov partition remain outside a fixed neighbourhood of the origin in the limit $N \to \infty$, the Gibbs measure is concentrated at this point approaching a δ distribution. For this reason the phase is dominated by the laminar points and is called laminar or non–hyperbolic phase.

Let us close this section by a brief view on the spectrum of the transfer operator (21) in the limit of large N. Beside the real eigenvalue analysed above one can show that the remaining $2N+1$ solutions of eq.(22) accumulate on the unit circle (appendix B) leading to a continuous part of the spectrum in the limit $N \to \infty$. The phase transition discussed above is therefore accomplished by a degeneracy of a single eigenvalue with a continuous part of the spectrum. Such a behaviour has also been observed in different non–hyperbolic systems [8, 22]. The fact that in the hyperbolic phase the system admits only one discrete eigenvalue is based on the very simple structure of the repeller in the region $x > 0$ which possesses no intrinsic frequency. Different models that means differnt choices of the c_k for low k values can introduce additional discrete eigenvalues. Of course the nature of the continuous part of the spectrum is not affected, as it is determined by the asymptotic behaviour for large k values.

4 Power spectrum

As the Gibbs measure changes drastically during the phase transition a similar change in the power spectrum is expected reflecting the time correlations in both phases. To study the power spectrum of our intermittent model system we can easily use eq.(17) respectively its matrix representation (cf. appendix A). In the case of our special function $u(x)$ it reads [7]

$$\begin{aligned} I_q(\omega) &= \{J_q(\omega) + J_q(-\omega) - 1\}\mu_{N+1} \\ J_q(\omega) &= \left(\frac{1}{1 - \frac{e^{i\omega}}{\lambda_q^{(0)}} \underline{\underline{H}}^u_q} \right)_{N+1,N+1} \quad . \end{aligned} \tag{31}$$

The resolvent governing eq.(31) can be computed easily as the structure of the transfer matrix (21) is rather simple. After a straightforward calculation we end up with (Appendix C)

$$J_q(\omega) = \frac{1}{1 - \alpha^{(N)} e^{q-q_c} \left\{ \sum_{k=0}^{N} c_k \left(\frac{e^{i\omega}}{\lambda_q^{(0)}} \right)^{2N+2-k} + \sum_{k=0}^{N} c_k \left(\frac{e^{i\omega}}{\lambda_q^{(0)}} \right)^{k+1} \right\}} \quad . \tag{32}$$

We will analyse this expression in the hyperbolic and non–hyperbolic phase.

i) Hyperbolic phase, $q > q_c, N \to \infty$: As $\lambda_q^{(0)} > 1$ in the limit of large N the first sum in eq.(32) vanishes in this limit. Hence we obtain

$$J_q(\omega) = \frac{1}{1 - e^{q-q_c} g(e^{i\omega}/\lambda_q^{(0)})} \tag{33}$$

[7] $(\cdots)_{N+1,N+1}$ denotes the lower right matrix element.

where the abbreviation (27) has been used. $\lambda_q^{(0)}$ is determined by the solution of eq.(26). The power spectrum is given by eq.(31) with μ_{N+1} determined by eq.(30). Our result (33) shows that the spectrum is a smooth function of ω without a strong frequency dependence. Especially near the phase transition point $q \gtrsim q_c$ the expression can be evaluated easily using the relation (29). Referring back to eq.(30) we obtain immediately

$$\mu_{N+1} \simeq -\frac{1}{\ln(\lambda_q^{(0)}-1)} \quad . \tag{34}$$

To evaluate eq.(33) we have to distinguish between two frequency regions:
$\omega \gg \lambda_q^{(0)}-1$: The limits $q \downarrow q_c$, $\lambda_q^{(0)} \downarrow 1$ can be taken in eq.(33) as the expression remains regular. Using $\ln(1-e^{i\omega}) = \ln|2\sin\frac{\omega}{2}| + i(\omega-\pi)/2$, $\omega \in (0,\pi)$ and inserting eq.(33) into eq.(31) we end up with

$$I_q(\omega) \simeq \left\{ \frac{(1-\cos\omega)\ln|2\sin\frac{\omega}{2}| + \frac{\pi-\omega}{2}\sin\omega}{(1-\cos\omega)\{(\ln|2\sin\frac{\omega}{2}|)^2 + (\frac{\pi-\omega}{2})^2\}} - 1 \right\} \mu_{N+1} \quad . \tag{35}$$

In the low frequency region $\pi \gg \omega \gg \lambda_q^{(0)}-1$ this expression reduces to $\pi/(-\ln(\lambda_q^{(0)}-1)\omega(\ln\omega)^2)$ showing a ω^{-1} decay modified by logarithmic contributions. But this behaviour is mainly suppressed by the small prefactor.
$\omega = O(\lambda_q^{(0)}-1)$: Inserting the abbreviation $\omega =: (\lambda_q^{(0)}-1)\hat{\omega}$ and the asymptotic result $q-q_c \simeq -(\lambda_q^{(0)}-1)\ln(\lambda_q^{(0)}-1)$ (cf. eq.(29)) into eq.(33) and equating the leading order in the small quantity $\lambda_q^{(0)}-1$ we obtain after a straigthforward expansion

$$I_q(\omega) \simeq -\frac{1}{(\lambda_q^{(0)}-1)(\ln(\lambda_q^{(0)}-1))^3} \left\{ -\frac{\ln(1+\hat{\omega}^2)}{\hat{\omega}^2} - 2\frac{\Omega(\hat{\omega})}{\hat{\omega}} \right\} \tag{36}$$

where $\Omega(\hat{\omega}) := \arg(1-i\hat{\omega})$, $\Omega \in (-\pi,\pi)$. This result shows clearly that the spectrum consists of a single line centered at $\omega = 0$. The linewidth is roughly given by $\lambda_q^{(0)}-1$ whereas the behaviour for large $\hat{\omega}$ shows again the ω^{-1} decay.

ii) Non–hyperbolic phase, $q < q_c$, $N \to \infty$: Now $\lambda_q^{(0)} \to 1$ in the limit of large N. Using the asymptotic relation $\lambda_q^{(0)} \simeq 1 - \ln(\exp(q_c - q) - 1)/(2N)$ which can be obtained from the eigenvalue equation (22), eqs.(25) and (32) simplify in the leading order of N (cf. appendix B)

$$\mu_{N+1} \simeq \frac{1}{2N(1-\exp(q-q_c))} \tag{37}$$

$$J_q(\omega) \simeq \frac{1}{1 - e^{q-q_c}\{(e^{q_c-q}-1)e^{i\omega(2N+3)}g(e^{-i\omega}) + g(e^{i\omega})\}} \tag{38}$$

where g denotes again the series (27). Inserting the expression (38) into eq.(31) we obtain the asymptotic expansion of the power spectrum for large N. The denominator of this equation changes rapidly with the period $2\pi/(2N) \simeq 2\pi/\tau$ where τ denotes the mean length of the laminar motion. This behaviour leads to a line spectrum, whose positions are separated by $2\pi/\tau$ (cf. Fig.3a). We have confirmed this qualitative argument by an elementary but tedious discussion of the expression (38) (cf. appendix C). From the same analysis we obtain that the envelope of the line spectrum obeys in the small frequency region the following relations for small, intermediate and large $q_c - q$ values

$$\begin{array}{ll} \mu_{N+1}\dfrac{4}{\pi\omega\{1+(2\ln\omega/\pi)^2\}} & 1/N \ll q_c - q \ll \omega \ll \pi \\ \mu_{N+1}\dfrac{4}{\pi\omega} & 1/N \ll \omega \ll (q_c-q)^{-1}, q_c - q \\ \mu_{N+1}\dfrac{4}{\pi\omega} & 1/N \ll (q_c-q)^{-1} \ll \omega \ll \pi \quad . \end{array} \tag{39}$$

In all cases an ω^{-1} power law decay of the envelope with logarithmic corrections is observed. Therefore the power spectrum in the non–hyperbolic phase shows the same features as the ordinary power spectrum ($q = 0$). Nevertheless we want to stress that the width of the single spectral lines decrease with decreasing q leading to a δ line spectrum in the limit $q \to -\infty$. This is in total accordance with our discussion of Gibbs measures, as in this limit the Gibbs measure is carried by the unstable periodic orbit which constitutes the Markov partition (cf. eq.(48)).

We confirm our analysis by showing numerical results of eqs.(22), (25), (31) and (32) for $N = 200$ and q in the hyperbolic and non–hyperbolic phase (Fig.3). Similar results are obtained for other values of the bifurcation parameter N. In the hyperbolic phase the spectrum mainly decreases with increasing frequency which means that the corresponding correlation function decays rapidly. This behaviour originates from the highly uncorrelated dynamics on the repeller which carries the Gibbs measure. In contrast the power spectrum in the non–hyperbolic phase shows a remarkable line structure in frequency space indicating an oscillating long time tail in the correlation function. The Gibbs measure is dominated by laminar motions which are interrupted by intermittent bursts causing the power law decay of the power spectrum. As q is decreased the Gibbs measure is mainly determined by the long time periodic orbits which results in a sharpening of the spectral lines. $>$ Fig.3

5 Phase transition region

In this section we want to investigate the transition between the laminar and the turbulent phase in more detail. To this end let us consider the immediate vicinity of the phase transition point q_c in the case of large but finite N. It is well known that under these circumstances scaling relations for the topological pressure occur which characterize the bifurcation [25]. Also for type I intermittent systems such scaling relations have been found recently [26]. In the first part of this section we want to consider these relations for our model. Afterwards we focus on the discussion of the power spectra in the scaling region. Although we have the explicit expressions (22), (25) (31) and (32) for the quantities of interest it is extremely difficult to analyse them analytically. We are therefore mainly forced to use numerical solutions of these equations.

As shown by Fig.2c the second derivative $\chi(q)$ admits a peak at $q_{max} = q_{max}(N)$ which turns left and becomes narrow as N increases. In order to establish the scaling relation we introduce a rescaled q variable in such a way that the peaks occur at the same position and with the same curvature in the maximum. With the abbreviation $\Delta q_{max} = q_{max} - q_c$ we define

$$q =: q_c + \Delta q_{max} + \left(-\frac{\chi(q_{max})}{\chi''(q_{max})}\right)^{1/2} x \quad . \tag{40}$$

Numerically we have confirmed the following scaling relations of the characteristic function and its derivatives with respect to the new variable x

$$\begin{aligned}
\Phi(q) &= \Phi(q_{max}) + \chi(q_{max}) \left(-\frac{\chi(q_{max})}{\chi''(q_{max})}\right) \tilde{\Phi}(x) \\
\langle u \rangle(q) &= \chi(q_{max}) \left(-\frac{\chi(q_{max})}{\chi''(q_{max})}\right)^{1/2} \tilde{u}(x) \\
\chi(q) &= \chi(q_{max}) \tilde{\chi}(x) \quad .
\end{aligned} \tag{41}$$

Here the scaling functions $\tilde{\Phi}$, $\tilde{u}$ and $\tilde{\chi}$ are defined by the right hand side. Fig.4 shows the result for N changing over more than one order in magnitude. A clear scaling behaviour has been obtained for $x < 1$. It does not hold in the whole x region which is also known from previous results but has not been fully understood yet. Fig.5 shows the N dependence of the scaling parameters Δq_{max}, $(-\chi(q_{max})/\chi''(q_{max}))^{1/2}$, $\chi(q_{max})$ and $\Phi(q_{max})$ which indicates an exponential dependence of the form $\sim N^{\delta}$ with exponents -0.72, -0.84, 0.64 respectively -0.89. > Fig.4 > Fig.5

Let us now turn to the discussion of the power spectra in the phase transition region. In view of our results presented above it seems reasonable to analyse the spectra in dependence of x instead of q. Furthermore having the relation $\lim_{\omega \to 0} I_q(\omega) = \chi(q)$ (appendix C) in mind we will normalize the spectra according to $\chi(q)$. Finally we refer back to the fact that below the phase transition point the spectrum consists of lines at intervals $2\pi/(2N)$. This suggests a rescaling of the frequency according to $\tilde{\omega} = N\omega/\pi$. In view of these facts we define a rescaled spectrum $\tilde{I}$

$$\tilde{I}(x; N\omega/\pi) := I_q(\omega)/\chi(q) \tag{42}$$

depending on the rescaled q value x. Fig.6 shows the spectra which have been obtained from eqs.(22), (25), (31) and (32) for $N = 200, 400, 800$ and several x values. First of all we mention that for negative x values line spectra occur. The position of the lines is independent of N indicating the correct frequency scaling in eq.(42). These spectra show strong resemblance to the spectra in the laminar phase (cf. section 4). As x changes to larger values the lines gradually disappear leading to a monotonic decreasing spectrum at $x \approx 1$ which marks roughly the end of the scaling region. Even if the rescaled spectra are obviously N dependent in the low frequency region we stress that in an intermediate frequency domain $\pi/N \ll \omega \ll \pi$ the envelope of the maxima depends only weakly on N and obeys a power law behaviour $\omega^{-\Theta_x}$ (cf. Fig.6a$_2$,6b$_2$). Fig.7 illustrates that the exponent Θ_x changes from -1 at extreme negative x values to about -0.6 at the end of the scaling region. The spectra for x larger than unity are monotonously decreasing (cf. Fig.6) and show strong similarity to the turbulent power spectrum investigated in the last section. It > Fig.6 > Fig.7

is surprising that these spectra are nearly N independent for small frequencies in the region $1 \lesssim x < 20$ although the topological pressure fails to obey a scaling behaviour (cf. Fig.4). This scaling property of the turbulent spectra is improved if x increases. It seems not to be related to the convergency of the power spectra towards a unique limit (cf. section 4) as the scaling property is valid in dependence on x instead of q. This fact may reflect some new universality in the time correlations of the Gibbs measure which emerges sligthly above the phase transition point. But at the moment we have no explanation at the hand.

6 Concluding remarks

Let us close with a few remarks on the observability of our results. In the laminar phase $q < q_c$ and especially at $q = 0$ (ordinary power spectrum) a line spectrum should be expected. But for two reasons it seems to be diffucult to observe this kind of spectrum. On the one hand this phase is dominated by nearly degenerated eigenvalues $\lambda_q^{(0)} \gtrsim |\lambda_q^{(l)}|$ where a quasi continuous part of the spectrum is involved. Therefore the expressions for the topological pressure and the power spectrum $I_q(\omega)$ converge very slowly towards the limit $n \to \infty$. On the other hand if the datasets contain noisy data the fine structure of the power spectrum is expected to be smeared out. For this reason some kind of finite frequency averaged spectrum $\int_{\Delta\omega} I_q(\omega')d\omega'/\Delta\omega$ will be observed where $\Delta\omega$ is of the order of the line distance. From eqs.(31) and (32) we obtain in the same approximation used in appendix C

$$\frac{1}{\Delta\omega}\int_{\Delta\omega} I_q(\omega')d\omega' \simeq \mu_{N+1}\left(2\mathrm{Re}\frac{1}{1-e^{q-q_c}g(e^{i\omega})}-1\right), \qquad (q < q_c) \quad . \tag{43}$$

This expression yields a smooth spectrum lacking a power law decay for small frequencies. In contrast to this an exponentially fast convergency in n is expected above the phase transition point ($q > q_c$) as the eigenvalues are well separated. This behaviour is far from being surprising because the hyperbolic phase is dominated by an axiom A like repeller. Therefore the power spectrum might be obtained for relatively small values of n quite accurate. It will yield by direct evaluation of the time series $u(T^i(x))$ information about the frequencies and damping rates associated with the chaotic repeller that dominates the turbulent bursts. The investigation of the phase transition region seems to be too difficult at the moment as it would require very large data sets due to the slow convergency of the quantities. To achieve this goal special methods for improving the convergency must be developed, which can deal with the nearly degenerated eigenvalue spectrum arising from the non–hyperbolicity of the system.

Nevertheless the anaysis of our simple model system demonstrates clearly that the thermodynamical approach is very useful to characterize dynamical systems also from the experimental point of view. For the evaluation of the quantities (4) and (6) only the knowledge of a time series and no underlying model system is required. The occurence of different phases may lead to considerable information about the dynamical process as they can be interpreted in terms of different invariant sets of the system. For this reason the application of this method to real data sets seems to be very fruitful.

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Appendix A

Consider a piecewise linear Markov map T with slope $\gamma_i = |T'_{I_i}|$ on the intervals I_i of the Markov partition. Let A_{ij} denote the transition matrix which is defined by $A_{ij} = 1$ if $T(I_i) \supseteq I_j$ and $A_{ij} = 0$ otherwise. Furthermore we presuppose that the function u is piecewise constant that means $u(x) = u_i$, $x \in I_i$. Then the space of functions spanned by the characteristic functions $\chi_i(x)$ of the intervals I_i is invariant under the action of the transfer operator (5). The latter admits with respect to the characteristic functions the matrix representation [20]

$$(\underline{\underline{H}}^u_q)_{ji} = \frac{e^{qu_i}}{\gamma_i} A_{ij} \quad . \tag{44}$$

The right eigenvectors $h_k^{(l)}$ of this matrix yield the eigenfunctions of the transfer operator via the relation $h^{(l)}(x) = \sum_k h_k^{(l)} \chi_k(x)$.

To obtain the explicit expression for the eigenmeasures $\nu^{(l)}$ apply the eigenvalue equation (cf. eq.(2)) to the characteristic function $\chi_k(x)$,

$$\begin{aligned} \lambda_q^{(l)} \nu^{(l)}(I_k) &= \lambda_q^{(l)} \int \chi_k(x) d\nu^{(l)} = \int \chi_k(x) d(\mathcal{H}_q^{u\dagger} \nu^{(l)}) \\ &= \int (\mathcal{H}_q^u \chi_k)(x) d\nu^{(l)} = \int \sum_j (\underline{\underline{H}}^u_q)_{jk} \chi_j(x) d\nu^{(l)} = \sum_j \nu^{(l)}(I_j) (\underline{\underline{H}}^u_q)_{jk} \quad . \end{aligned} \tag{45}$$

Therefore the $\nu^{(l)}$–measure of the intervals I_k of the Markov partition $\nu_k^{(l)} := \nu^{(l)}(I_k)$ can be obtained from the left eigenvector of the matrix (44). To get the full expression for the eigenmeasure consider the dynamic partition which is defined via the intervals

$$U_{i_0,\ldots,i_n} := \{x \mid T^k(x) \in I_{i_k}, 0 \le k \le n\} \quad . \tag{46}$$

Let $\nu^{(l)}_{i_0,\ldots,i_n}$ and $\chi_{i_0,\ldots,i_n}$ denote the $\nu^{(l)}$–measure respectively the characteristic function of these intervals. By the same reasoning as above we obtain

$$\lambda_q^{(l)} \nu^{(l)}_{i_0,\ldots,i_n} = \int (\mathcal{H}_q^u \chi_{i_0,\ldots,i_n})(x) d\nu^{(l)} = \int \frac{e^{qu_{i_0}}}{\gamma_{i_0}} \chi_{i_1,\ldots,i_n}(x) d\nu^{(l)} = \frac{e^{qu_{i_0}}}{\gamma_{i_0}} \nu^{(l)}_{i_1,\ldots,i_n} \quad . \tag{47}$$

Therefore the relation

$$\nu^{(l)}_{i_0,\ldots,i_n} = \prod_{k=0}^{n-1} \frac{e^{qu_{i_k}}}{\gamma_{i_k}} \cdot \frac{\nu^{(l)}_{i_n}}{(\lambda_q^{(0)})^n} \tag{48}$$

holds. Due to the prefactor this measure is in general multifractal reflecting the strange geometric structure of the chaotic set in contrast to the eigenfunctions which are smooth.

Using the formulas presented above eq.(17) for the power spectrum can be easily written down in the matrix representation. As $u \cdot h^{(0)}$ is a piecewise constant function eq.(44) can be applied to evaluate the resolvent $(1-\mathcal{H}_u^q e^{i\omega}/\lambda_q^{(0)})^{-1}$. We obtain

$$\begin{aligned} I_q(\omega) &= \int u(x) \sum_{k,j} \left(\left[1 - \frac{e^{i\omega}}{\lambda_q^{(0)}} \underline{\underline{H}}^u_q \right]^{-1} + \left[1 - \frac{e^{-i\omega}}{\lambda_q^{(0)}} \underline{\underline{H}}^u_q \right]^{-1} - 1 \right)_{kj} u_j h_j^{(0)} \chi_k(x) d\nu^{(0)} \\ &= \sum_{k,j} \nu_k^{(0)} u_k \left(\left[1 - \frac{e^{i\omega}}{\lambda_q^{(0)}} \underline{\underline{H}}^u_q \right]^{-1} + \left[1 - \frac{e^{-i\omega}}{\lambda_q^{(0)}} \underline{\underline{H}}^u_q \right]^{-1} - 1 \right)_{kj} u_j h_j^{(0)} \quad . \end{aligned} \tag{49}$$

Appendix B

Asymptotic behaviour of $\lambda_q^{(0)}$ for $q < q_c$: With the abbreviation

$$g_N(z) := \alpha^{(N)} \sum_{k=0}^{N} c_k z^{k+1} \tag{50}$$

the characteristic equation (22) reads

$$1 = e^{q-q_c} \{ (\lambda_q^{(0)})^{-(2N+3)} g_N(\lambda_q^{(0)}) + g_N(1/\lambda_q^{(0)}) \} \quad . \tag{51}$$

Suppose that the relation

$$\lambda_q^{(0)} =: 1 + \frac{a_q}{N} \tag{52}$$

holds with $a_q = O(1)$ in the limit of large N. Then $\exp(-|a_q|) \leq (\lambda_q^{(0)})^k \leq \exp(|a_q|)$ for $0 \leq k \leq N$ and $(\lambda_q^{(0)})^k \to 1$ for fixed k and $N \to \infty$. Hence for arbitrary but fixed $M \leq N$ the relation

$$g_N(\lambda_q^{(0)}) = \alpha^{(N)} \sum_{k=0}^{M} c_k (1 + \frac{a_q}{N})^k + \alpha^{(N)} \sum_{k=M+1}^{N} c_k (1 + \frac{a_q}{N})^k \tag{53}$$

shows that $g_N(\lambda_q^{(0)})$ tends towards $\alpha^{(\infty)} \sum_{k=0}^{M} c_k + O(1/M)$ as N goes to infinity. By choosing M arbitrarily large we obtain

$$g_N(\lambda_q^{(0)}) \to 1, \qquad (N \to \infty) \quad . \tag{54}$$

As the same reasoning holds for $g_N(1/\lambda_q^{(0)})$ also eq.(51) yields in the limit of asymptotic large N

$$1 = e^{q-q_c} (e^{-2a_q} + 1), \qquad (N \to \infty) \quad . \tag{55}$$

Finally eq.(52) leads to

$$\lambda_q^{(0)} \simeq 1 - \frac{\ln(\exp(q_c - q) - 1)}{2N} \tag{56}$$

which is proven easily to be the correct asymptotic behaviour.

Asymptotic behaviour of μ_{N+1} for $q < q_c$: Rewriting eq.(25) in the form

$$\begin{aligned} 1 &= e^{q-q_c}\alpha^{(N)}\{(2N+3)(\lambda_q^{(0)})^{-(2N+3)}\sum_{k=0}^{N} c_k(\lambda_q^{(0)})^{k+1} \\ &\quad +(1-(\lambda_q^{(0)})^{-(2N+3)})\sum_{k=0}^{N}(k+1)c_k(\lambda_q^{(0)})^{k+1}\}\mu_{N+1} \end{aligned} \tag{57}$$

and using the same reasoning as above we get

$$1 = e^{q-q_c}\{2Ne^{-2a_q} + O(\ln N)\}\mu_{N+1} \quad . \tag{58}$$

Eqs.(55) and (58) yield the asymptotic result (37).

Asymptotic behaviour of $J_q(\omega)$ for $q < q_c$: By the same argument that has us led to eq.(54) we obtain

$$g_N(e^{i\omega}/\lambda_q^{(0)}) \to g(e^{i\omega}), \qquad g_N(\lambda_q^{(0)}/e^{i\omega}) \to g(e^{-i\omega}), \qquad (N\to\infty) \quad . \tag{59}$$

As in view of eq.(52) $(\lambda_q^{(0)})^{2N+3}$ tends towards e^{-2a_q} the asymptotic expression (38) is obtained by taking eq.(55) into account.

Spectrum of $\underline{\underline{H}}_q^u$ in the limit $N \to \infty$: In a first step we will show that there exists no eigenvalue inside the complex unit circle. Then we show that sole the largest eigenvalue $\lambda_q^{(0)}$ may have absolute value larger than unity.

Suppose that for a fixed $\delta > 0$ a solution of the eigenvalue equation (51) obeys $\lambda_q \leq 1-\delta$. As

$$\begin{aligned} |g_N(1/\lambda_q)| &\leq \alpha^{(N)}\sum_{k=0}^{N} c_k|\lambda_q|^{-(N+1)} = |\lambda_q|^{-(N+1)} \\ |g_N(\lambda_q)| &= \alpha^{(N)}|c_0 + \sum_{k=1}^{N} c_k\lambda_q^k||\lambda_q| \geq \alpha^{(N)}(c_0 - \sum_{k=1}^{N} c_k)|\lambda_q| = \frac{|\lambda_q|}{N+1} \end{aligned} \tag{60}$$

we have

$$\begin{aligned} &|(\lambda_q)^{-(2N+3)}g_N(\lambda_q) + g_N(1/\lambda_q)| \geq \frac{|\lambda_q|^{-(2N+2)}}{N+1} - |\lambda_q|^{-(N+1)} \\ &\geq (1-\delta)^{-(N+1)}(\frac{(1-\delta)^{-(N+1)}}{N+1} - 1) \to \infty, \qquad (N\to\infty) \end{aligned} \tag{61}$$

which contradicts the eigenvalue equation (51).

As for $q < q_c$ the relation $\lambda_q^{(0)} \to 1$ holds in the limit $N \to \infty$ and $\lambda_q^{(0)}$ is the largest eigenvalue by Perron's theorem there remains nothing to prove. Therefore let us assume that $q > q_c$ and suppose that for fixed $\delta > 0$ there exists an eigenvalue $\lambda_q \neq \lambda_q^{(0)}$ obeying the relation $(1+\delta) \leq |\lambda_q| \leq \lambda_q^{(0)}$. From the eigenvalue equation (51) we obtain

$$\begin{aligned} 0 &= e^{q-q_c}\{\lambda_q^{-(2N+3)}g_N(\lambda_q) + g_N(1/\lambda_q) - (\lambda_q^{(0)})^{-(2N+3)}g_N(\lambda_q^{(0)}) - g_N(\lambda_q^{(0)})\} \\ &\geq e^{q-q_c}\{|g_N(1/\lambda_q) - g_N(1/\lambda_q^{(0)})| - |\lambda_q^{-(2N+3)}g_N(\lambda_q) - (\lambda_q^{(0)})^{-(2N+3)}g_N(\lambda_q^{(0)})|\} \end{aligned} \tag{62}$$

The inequality $|1-(\lambda_q/\lambda_q^{(0)})^{2N+2-k}| \leq |1-\lambda_q/\lambda_q^{(0)}|(2N+2-k)$ which can be proven easily by using the geometric sum yields the following estimate for the second term

$$|\lambda_q^{-(2N+3)} g_N(\lambda_q) - (\lambda_q^{(0)})^{-(2N+3)} g_N(\lambda_q^{(0)})| \leq \alpha^{(N)} \sum_{k=0}^{N} c_k |\lambda_q|^{-(2N+2-k)} |1-(\lambda_q/\lambda_q^{(0)})^{2N+2-k}|$$

$$\leq \alpha^{(N)} \sum_{k=0}^{N} c_k (2N+2-k)/\lambda_q^{(0)} \cdot (1+\delta)^{-N} |\lambda_q - \lambda_q^{(0)}| \leq K_0 (1+\delta)^{-N/2} |\lambda_q - \lambda_q^{(0)}| \qquad (63)$$

where K_0 denotes a N independent constant. For the first term of eq.(62) the inequality

$$|g_N(1/\lambda_q) - g_N(1/\lambda_q^{(0)})| \geq K_1 |\lambda_q - \lambda_q^{(0)}| \qquad (64)$$

is valid with some positive constant K_1 on the compact set $(1+\delta) \leq |\lambda_q| \leq \lambda_q^{(0)}$ of the complex λ_q plane. This relation follows from the observation that eq.(26) has a unique solution and $g'(1/\lambda_q^{(0)}) \neq 0$ holds. As a consequence $|g(1/\lambda_q) - g(1/\lambda_q^{(0)})|/|\lambda_q - \lambda_q^{(0)}|$ is strictly positive. This property carries over to finite but sufficiently large N as the power series converges uniformly. Eqs.(62), (63) and (64) yield now

$$0 \geq (K_1 - K_0(1+\delta)^{-N/2}) |\lambda_q - \lambda_q^{(0)}| \qquad (65)$$

which contradicts for sufficiently large N the assumption $\lambda_q \neq \lambda_q^{(0)}$.

The eigenvalues except $\lambda_q^{(0)}$ accumulate in a band around the unit circle. From our derivation the width of this band can be estimated to $O(1/N)$.

Appendix C

Derivation of eq.(32): For computing the resolvent (31_2) we solve the algebraic equation

$$(1 - z\underline{\underline{H}}^u_q)\underline{x} = \underline{u} \qquad (66)$$

for $\underline{x}$ where $u_i = 0$ for $-N \leq i \leq N$, $u_{N+1} = 1$ and $z := e^{i\omega}/\lambda_q^{(0)}$. The desired matrix element (31_2) is then given by x_{N+1}. In view of the expression (21) eq.(66) reads

$$\begin{aligned} x_{-N} - e^q \frac{z}{\gamma_{N+1}^{(N)}} x_{N+1} &= 0 \\ -\frac{z}{\gamma_k^{(N)}} x_k + x_{k+1} - e^q \frac{z}{\gamma_{N+1}^{(N)}} x_{N+1} &= 0, \qquad -N \leq k \leq N-1 \\ -\frac{z}{\gamma_N^{(N)}} x_N + \left(1 - e^q \frac{z}{\gamma_{N+1}^{(N)}}\right) x_{N+1} &= 1 \quad . \end{aligned} \qquad (67)$$

Eq.(67_2) leads to

$$x_k = \prod_{l=-N}^{k-1} \frac{z}{\gamma_l^{(N)}} \cdot x_{-N} + \sum_{m=-N}^{k-1} \prod_{l=m+1}^{k-1} \frac{z}{\gamma_l^{(N)}} \cdot \frac{z}{\gamma_{N+1}^{(N)}} e^q x_{N+1} \quad . \qquad (68)$$

Inserting eq.(67_1) into eq.(68) and equating the expression for $k = N$ we obtain from eq.(67_3)

$$x_{N+1}\left\{1 - e^q \sum_{k=-N}^{N+1} \prod_{l=k}^{N+1} \frac{z}{\gamma_l^{(N)}}\right\} = 1 \quad . \tag{69}$$

With the definition of $\gamma_l^{(N)}$ (cf. section 3) we have $\prod_{l=k}^{N+1} \gamma_l^{(N)} = 2/\Delta a_k^{(N)}$. Hence eq.(69) reads by taking the definitions (18) and (28) into account

$$x_{N+1}\left\{1 - e^{q-q_c}\left(\sum_{k=0}^{N} z^{2N+2-k}\alpha^{(N)}c_k + \sum_{k=0}^{N} z^{k+1}\alpha^{(N)}c_k\right)\right\} = 1 \tag{70}$$

which is the desired result (32).

Shape of $I_q(\omega)$ for $q < q_c$, $N \to \infty$: Introducing the abbreviation

$$r(\omega)e^{i\phi(\omega)} := e^{q-q_c}g(e^{i\omega}), \qquad r(\omega), \phi(\omega) \in \mathbf{R} \tag{71}$$

the denominator of $J_q(\omega)$ reads (cf. eqs.(38) and (55))

$$Z(\omega) := 1 - r(\omega)(e^{-2a_q}e^{i(\omega(2N+3)-\phi(\omega))} + e^{i\phi(\omega)}) \quad . \tag{72}$$

The power spectrum (31_1) in terms of this quantity is given by

$$I_q(\omega) = \mu_{N+1}(\frac{Z(\omega) + Z^*(\omega)}{|Z(\omega)|^2} - 1) \quad . \tag{73}$$

Hence its extrema are determined by the condition

$$0 = Z'(\omega)(Z^*(\omega))^2 + Z^{*\prime}(\omega)(Z(\omega))^2 \quad . \tag{74}$$

To the leading order in N the derivative of the quantity (72) can be calculated by keeping the slowly varying functions $r(\omega)$ and $\phi(\omega)$ constant,

$$Z'(\omega) = -i(2N+3)e^{-2a_q}r(\omega)e^{i(\omega(2N+3)-\phi(\omega))} + O(1) \quad . \tag{75}$$

Inserting eqs.(72) and (75) into the condition (74) we obtain after a little algebra

$$\sin(\omega(2N+3) + \alpha(\omega) - \phi(\omega)) = -\frac{2r^2(\omega)e^{-2a_q}}{R(\omega)}\sin\phi(\omega) \tag{76}$$

where the abbreviation

$$R(\omega)e^{i\alpha(\omega)} := r^2(\omega)e^{-4a_q} - (1 - r(\omega)e^{-i\phi(\omega)})^2, \qquad R(\omega), \alpha(\omega) \in \mathbf{R} \tag{77}$$

has been used. Eq.(76) is the desired implicit equation to determine the maxima of the power spectrum [8]. As the quantities $r(\omega)$, $\phi(\omega)$, $R(\omega)$ and $\alpha(\omega)$ vary on a frequency scale of the

[8] Using the definition (77) it can be shown that the right hand side of eq.(76) has an absolute value less than 1.

order $O(1)$ they can be treated as constant if frequency intervals of the order $O(1/N)$ are considered. Hence we obtain that eq.(76) admits two solutions in every frequency interval of length $\Delta\omega = 2\pi/(2N+3)$ corresponding to a maximum and a minimum value of the spectrum. The latter consists of single lines which are separated by $\Delta\omega \simeq 2\pi/(2N)$.

To obtain the envelope we solve eq.(76) for $\exp(i[\omega(2N+3)-\phi(\omega)])$ and obtain for the positions of the maxima

$$\begin{aligned} e^{i(\omega(2N+3)-\phi(\omega))} &= R(\omega)e^{-i\alpha(\omega)}\frac{-i2r^2(\omega)e^{-2a_q}\sin\phi(\omega)+\sqrt{R^2(\omega)-(2r^2(\omega)e^{-2a_q}\sin\phi(\omega))^2}}{R^2(\omega)} \\ &= 1-i\frac{2r(\omega)\sin\phi(\omega)}{r(\omega)e^{-2a_q}+1-r(\omega)e^{-i\phi(\omega)}} \end{aligned} \tag{78}$$

where the last expression follows by tedious algebra. Insertion into eq.(72) yields for the envelope (73) the result[9]

$$(I_q(\omega))_{env.} \simeq \mu_{N+1}\left(2\frac{1-e^{q-q_c}\mathrm{Re}g(e^{i\omega})+(1-e^{q-q_c})|g(e^{i\omega})|}{1-2e^{q-q_c}\mathrm{Re}g(e^{i\omega})-(1-2e^{q-q_c})|g(e^{i\omega})|^2}-1\right) \quad . \tag{79}$$

The small frequency expansion (cf. eq.(27))

$$g(e^{i\omega}) = 1+\omega\frac{\omega-\pi}{2}-\frac{1}{2}\omega^2\ln\omega-i\omega\ln\omega+i\frac{\pi}{4}\omega^2+O(\omega^3\ln\omega), \qquad \omega>0 \tag{80}$$

leads to the result (39).

Spectrum in the limit of small frequencies: As long as $\langle u\rangle(q)=\int u d\tilde{\mu}_q$ does not vanish the power spectrum contains a δ contribution at $\omega=0$ because the correlation function (7) does not decay to zero. In contrast the power spectrum evaluated with the function $\delta u(x):=u(x)-\langle u\rangle(q)$ is continuous at $\omega=0$ but otherwise unchanged. Therefore by eq.(6_2)

$$\begin{aligned} \lim_{\omega\to 0} I_q(\omega) &= \lim_{\omega\to 0}\lim_{n\to\infty}\langle\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\delta u(T^k(x))e^{-i\omega k}\right|^2\exp(qnU_n(x))\rangle/\langle\exp(qnU_n(x))\rangle \\ &= \lim_{n\to\infty}\langle\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\delta u(T^k(x))\right|^2\exp(qnU_n(x))\rangle/\langle\exp(qnU_n(x))\rangle = \chi(q) \quad . \end{aligned} \tag{81}$$

The last equality follows directly by differentating eq.(4) twice.

[9] We obtain a smooth function for the envelope as the line distance and the accuracy of or asymptotic expansion are both of the order $O(1/N)$.

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Figure captions

Fig.1: Type I intermittent piecewise linear model map for $N = 3$. $I_k^{(N)}$, $-N \le k \le N+1$ indicate the intervals of the Markov partition. The length of the laminar motion N acts as the bifurcation parameter.

Fig.2: Topological pressure $\Phi(q)$ and its derivatives $\langle u\rangle(q)$ and $\chi(q)$ for $N = 25$ ( ), $N = 50$ ( ) and $N = 200$ ( ).

Fig.3: Power spectrum $I_q(\omega)$ for $N = 200$ and

a) $q = -2$ ( ), $q = 0$ ( ), $q = 0.6$ ( ) (non–hyperbolic phase)

b) $q = 0.75$ ( ), $q = 0.77$ ( ), $q = 0.79$ ( ) (hyperbolic phase)

c) Maxima of the spectrum for $q = -2$ ( ), $q = 0$ ( ), $q = 0.6$ ( ) (cf. a)).

Fig.4: Rescaled topological pressure $\tilde{\Phi}(x)$ and its derivatives $\tilde{u}(x)$ and $\tilde{\chi}(x)$ for $N = 25$ ( ), $N = 50$ ( ), $N = 200$ ( ) and $N = 800$ ( ).

Fig.5: N dependence of the scaling parameters (A) Δq_{max}, (B) $(-\chi(q_{max})/\chi''(q_{max}))^{1/2}$, (C) $\chi(q_{max})$ and (D) $\Phi(q_{max})$.

Fig.6: Rescaled power spectrum $\tilde{I}(x; N\omega/\pi)$ and its maxima for $N = 200$ (upper line,), $N = 400$ (middle line,) and $N = 800$ (lower line,).

a$_1$) Spectrum for $x = -20$

a$_2$) Maxima for $x = -20$

b$_1$) Spectrum for $x = -2$

b$_2$) Maxima for $x = -2$

c) Spectrum for $x = 0.75$

d) Spectrum for $x = 2.5$

Fig.7: Maxima of the power spectrum for $N = 800$ and several x values in an intermediate frequency region. $x = -20$ ( ), $x = -10$ ( ), $x = -5$ ( ), $x = -2$ ( ), $x = 0$ ( ) and $x = 0.75$ ( ).