Projection Operator Approach to the Thermodynamical Formalism of Dynamical Systems

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Abstract

An analytical perturbative treatment of characteristic exponents describing the fluctuations of temporal coarse grained quantities in the context of nonlinear dynamical systems is proposed. It is based on the analysis of the resolvent of the corresponding transfer operator by a projection operator method similar to those used in statistical mechanics. Two different approximation schemes are presented and tested for the case of an exactly solvable but nontrivial model system.

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1 Introduction

The thermodynamical formalism constitutes a program for analysing the complicated behaviour of nonlinear dynamical systems [1, 2]. The main quantity that contains a lot of information about the dynamics is given by the characteristic exponent [3]

$$\Phi^u(q) := \lim_{n\to\infty} \frac{1}{n} \ln\langle \exp\left(q \sum_{i=0}^{n-1} u(T^i(x)) \right)\rangle \tag{1}$$

where to be definit we want to restrict the discussion to discrete dynamical systems $x_{n+1} = T(x_n)$. The expectation value in eq.(1) $\langle \ldots \rangle$ is meant with respect to some distribution of initial points x which is usually assumed to be the (physical) invariant distribution (SRB measure in mathematical terms). The characteristic quantity depends on the function $u(x)$. The case that $u(x)$ is given by the local expansion rate is by Bowens theorem [2] of special importance and frequently discussed in the literature [4]. In those cases the quantity (1) is called the topological pressure. One possibility for the computation of the characteristic exponent is based on the introduction of the transfer operator [5]

$$(\mathcal{H}_q^u h)(x) := \int \delta(x - T(y)) \exp(qu(y)) h(y)\, dy \quad . \tag{2}$$

Its largest eigenvalue λ_q is connected to the characteristic exponent via the relation $\ln \lambda_q = \Phi^u(q)$. It is the aim of this paper to apply the ideas of the projection operator technique of statistical mechanics [6, 7] for an analytical perturbative calculation of the characteristic exponent.

In order to define a projection operator on the space of integrable functions its dual space has to be introduced in a natural way. Refering back to the representation of the SRB measure proposed by Bowen [2] it is obvious to consider the space of measures as the dual space and to introduce the bilinear expression

$$(\mu|h) := \int h\, d\mu \quad . \tag{3}$$

Then the SRB measure μ_{SRB} admits the representation $d\mu_{SRB} = h_* d\nu_*$ where h_* and ν_* denote the eigenelements of the Ruelle–Frobenius–Perron operator $\mathcal{L}$ and its formally adjoint [1] $\mathcal{L}^\dagger$ corresponding to the largest real eigenvalue. In systems admitting an attractor this eigenvalue equals 1, and for the special case of one dimensional maps ν_* coincides with the Lebesgue measure so that the SRB measure is determined by by the invariant distribution h_*. By this theorem the dual space structure and the definition (3) is closely related to the dynamical systems under consideration.

In general higher dimensional systems no relation between $\mathcal{H}_q^u$ and $\mathcal{L}$ is known to my knowledge. Although such a relation is not necessary for the formal investigations made in this article, it is important for the construction of an appropriate projection operator. For this reason we will concentrate on one dimensional systems [2] and expanding maps [8] where

[1] with respect to the bilinear expression (3)

[2] Precisely spoken also some expansive condition like $\exists N, \forall x, \prod_{i=0}^{N} |T'(T^{(i)}(x))| > 1$ has to be imposed.

the relation $\mathcal{H}^u_{q=0} = \mathcal{L}$ is well established. Also the function space on which the transfer operator is defined may depend on the properties of the dynamical system. In the case of one dimensional maps this space is given by piecewise continuous functions [3].

With these settings we will show in the next section how a projection operator formalism for the resolvent of the transfer operator (2) according to the lines of the famos Mori projection operator technique [7] can be constructed. The third section is devoted to the outline of an approximation scheme which is closely connected to the continued fraction expansion for correlation functions in equilibrium statistical mechanics [9] and yields a result that has been obtained earlier in a phenomenological way [10]. The fourth section contains an alternative approximation scheme which relies on a perturbation expansion of the memory kernel. Section 5 is devoted to the discussion of a simple exactly solvable model system from the viewpoint of these perturbation expansions. Finally the results will be summarized.

2 Projection Operator Technique for the Resolvent

For the determination of the eigenvalues of the transfer operator the discussion of the resolvent $(z - \mathcal{H}^u_q)^{-1}$ is a convenient tool. Its singularities define the spectrum of the transfer operator. With the help of the projection operator formalism this resolvent can be devided into several parts which allow for a perturbative treatment. For this reason consider a suitably chosen set of relevant functions $\{g_\alpha\}$ and measures $\{\nu_\beta\}$, $\alpha, \beta = 1, \ldots, N$ and define a "projection operator"

$$\mathcal{P} := \sum_{\alpha,\beta} |g_\alpha)(\underline{\underline{\chi}}^{-1})_{\alpha\beta}(\nu_\beta|, \qquad \chi_{\beta\alpha} := (\nu_\beta|g_\alpha) \tag{4}$$

on the space of integrable functions which admits the usual relations

$$\mathcal{P}^2 = \mathcal{P}, \qquad \mathcal{P} g_\alpha = g_\alpha, \qquad \mathcal{P}^\dagger \nu_\beta = \nu_\beta \quad . \tag{5}$$

Due to the fact that eq.(3) does not represent a scalar product on some Hilbert space the regularity of the matrix $\underline{\underline{\chi}}$ is not obvious. This property is based on an appropriate choice of the sets $\{g_\alpha\}$ and $\{\nu_\beta\}$.

Applying the well known operator identity [12, p.367]

$$\mathcal{P}[z - \mathcal{P}\mathcal{H}^u_q\mathcal{P} - \mathcal{P}\mathcal{H}^u_q(z - \mathcal{Q}\mathcal{H}^u_q\mathcal{Q})^{-1}\mathcal{Q}\mathcal{H}^u_q\mathcal{P}]\mathcal{P}(z - \mathcal{H}^u_q)^{-1}\mathcal{P} = \mathcal{P}, \quad \mathcal{Q} := 1 - \mathcal{P} \tag{6}$$

to the resolvent $(\nu_\beta|(z - \mathcal{H}^u_q)^{-1}|g_\alpha)$ one gets with the help of eqs.(4) and (5) the final result

$$(\nu_\beta|(z - \mathcal{H}^u_q)^{-1}|g_\alpha) = \left(\frac{1}{z\underline{\underline{1}} - \underline{\underline{\Omega}} - \underline{\underline{\Gamma}}(z)} \underline{\underline{\chi}} \right)_{\beta\alpha} \tag{7}$$

[3] Finite jumps may occur on the countable set of points at which T' is discontinuous and its forward iterates.

where the frequency matrix $\underline{\underline{\Omega}}$ and the memory kernel $\underline{\underline{\Gamma}}(z)$ are given by

$$\begin{aligned} \Omega_{\beta\alpha} &:= \sum_\gamma (\nu_\beta|\mathcal{H}_q^u|g_\gamma)(\underline{\underline{\chi}}^{-1})_{\gamma\alpha} \\ \Gamma_{\beta\alpha}(z) &:= \sum_\gamma (\nu_\beta|\mathcal{H}_q^u \mathcal{Q}\frac{1}{z-\mathcal{Q}\mathcal{H}_q^u\mathcal{Q}}\mathcal{Q}\mathcal{H}_q^u|g_\gamma)(\underline{\underline{\chi}}^{-1})_{\gamma\alpha} \quad . \end{aligned} \tag{8}$$

As mentioned above the poles of the resolvent (7) yield the eigenvalues of the transfer operator (2). Inspecting eq.(7) one recognizes that this problem has been reduced to the computation of the frequency and memory matrix and solving the algebraic equation $\det(z\underline{\underline{1}} - \underline{\underline{\Omega}} - \underline{\underline{\Gamma}}(z)) = 0$. We want to show in the next sections that the expressions (8) are more adapted to the application of analytical perturbation schemes than the original resolvent.

3 Continued Fraction Expansion

As one is mainly interested in the characterisrtic exponent (1) if the average is taken with respect to the natural invariant measure it is suggestive to choose the sets $\{g_\alpha\}$ and $\{\nu_\beta\}$ in such a way that they contain the SRB measure and investigate the poles of the resolvent

$$(\nu_*|\frac{1}{z-\mathcal{H}_q^u}|h_*) \quad . \tag{9}$$

For this reason let us take the smallest sets $\{g_1\} = \{h_*\}$ and $\{\nu_1\} = \{\nu_*\}$, $\alpha, \beta = 1$ which fit in with these constraint. Although eq.(7) yields an exact expression for the resolvent under consideration it is a hard task to compute the memory kernel $\underline{\underline{\Gamma}}(z)$ in general. Let us therefore trace back to an idea of H. Mori [9] and incorporate the higher order functions $(\mathcal{H}_q^u)^n h_*$ and measures $(\mathcal{H}_q^{u\dagger})^n \nu_*$ $(n = 1, \ldots, N)$ into the relevant set. By this procedure a continued fraction expansion for the resolvent (9) can be constructed.

To be definite take as the relevant sets those which are build up by $(\mathcal{H}_q^u)^n h_*$ and $(\mathcal{H}_q^{u\dagger})^n \nu_*$ $(n = 0, \ldots, N)$.

$$\begin{aligned} g_1 &:= h_* \\ \nu_1 &:= \nu_* \\ g_{i+1} &:= (\prod_{k=1}^{i} \mathcal{Q}_k)\mathcal{H}_q^u g_i, \quad \mathcal{Q}_k := 1 - \frac{|g_k)(\nu_k|}{(\nu_k|g_k)}, \quad i = 1, \ldots, N-1 \\ \nu_{i+1} &:= (\prod_{k=1}^{i} \mathcal{Q}_k^\dagger)\mathcal{H}_q^{u\dagger}\nu_i, \quad \mathcal{Q}_k^\dagger := 1 - \frac{|\nu_k)(g_k|}{(\nu_k|g_k)}, \quad i = 1, \ldots, N-1 \quad . \end{aligned} \tag{10}$$

To simplify the considerations we have chosen orthogonalized sets. In writing down eq.(10) it has been presupposed that the relations $(\nu_i|g_i) \neq 0$, $i = 1, \ldots, N$ hold. For a remark on the opposite case we refer to Appendix B. After a short computation (Appendix A) we obtain the following results for the frequency matrix

$$\underline{\underline{\Omega}} \;=\; \begin{pmatrix} \Omega_{11} & 1 & 0 & \cdots & 0 \\ \Omega_{21} & \Omega_{22} & \ddots & & \vdots \\ 0 & \Omega_{32} & \ddots & 1 & 0 \\ \vdots & & \ddots & \Omega_{N-1\,N-1} & 1 \\ 0 & \cdots & 0 & \Omega_{N\,N-1} & \Omega_{NN} \end{pmatrix}$$

$$\Omega_{i\,i} := \frac{(\nu_i|\mathcal{H}_q^u|g_i)}{(\nu_i|g_i)}, \quad \Omega_{i+1\,i} := \frac{(\nu_{i+1}|g_{i+1})}{(\nu_i|g_i)} \tag{11}$$

and the memory kernel

$$\underline{\underline{\Gamma}}(z) \;=\; \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 0 \\ 0 & \cdots & 0 & \Gamma_{NN}(z) \end{pmatrix}$$

$$\Gamma_{NN}(z) \;:=\; (\nu_N|\mathcal{H}_q^u\mathcal{Q}\frac{1}{z-\mathcal{Q}\mathcal{H}_q^u\mathcal{Q}}\mathcal{Q}\mathcal{H}_q^u|g_N) \quad . \tag{12}$$

Inserting these expressions into the general eq.(7) one gets [12, p.230] for the matrix element (9) the continued fraction expression

$$(\nu_1|\frac{1}{z-\mathcal{H}_q^u}|g_1) = \cfrac{1}{z-\Omega_{11}-\cfrac{\Omega_{21}}{z-\Omega_{22}-\cfrac{\Omega_{32}}{\ddots\,\cfrac{}{z-\Omega_{N-1\,N-1}-\cfrac{\Omega_{N\,N-1}}{z-\Omega_{NN}-\Gamma_{NN}(z)}}}}} \quad . \tag{13}$$

This expansion is mainly determined by the elements of the frequency matrix (11) that means by the moments of the transfer operator which can be computed easily when the SRB measure is known. It coincides with the expansion proposed by Fujisaka and Inoue [10] (Appendix B).

4 Perturbation Expansion of the Memory Kernel

Assume that the transfer operator can be devided into a zero order part $\mathcal{H}_q^0$ which allows for an explict treatment and a small perturbation $\mathcal{H}_q^1$. A perturbative treatment of the eigenvalue problem of the transfer operator along the lines of the Schrödinger perturbation theory used in quantum mechanics [4] is of limited value in the vicinity of q–values q_0 at which the operator (2) admits degenerated eigenvalues. Due to small denominators the radius of convergence of such an expansion decreases as $|q-q_0|$. It is true that around this critical

[4] For the perturbation series of non hermitian operators see e.g. [11].

point an expansion in both small quantities $q - q_0$ and $\mathcal{H}^1_q$ is possible, but no expression uniformly valid in q can be obtained by this perturbation scheme in any finite order.

In contrast to this it is known that a standard perturbation expansion of the quantities (8) yields a uniformly valid expansion of the resolvent (7) if the projection operator is chosen appropriately [6, 12]. In order to perform the formal expansion let us choose the relevant sets in such a way that the projection operator (4) commutes with the unperturbed part of the transfer operator

$$\mathcal{H}^u_q =: \mathcal{H}^0_q + \mathcal{H}^1_q, \qquad \mathcal{P}\mathcal{H}^0_q = \mathcal{H}^0_q\mathcal{P} \quad . \tag{14}$$

The last property can be achieved if one chooses some eigenelements of the zero order part as the relevant sets. As we are interested in the behaviour of the largest eigenvalue the eigenelements corresponding to the upper part of the spectrum seem to be the appropriate choice. By virtue of eq.(14) the definitions (8) read

$$\begin{aligned} \Omega_{\beta\alpha} &= \Omega^{(0)}_{\beta\alpha} + \Omega^{(1)}_{\beta\alpha} = \sum_\gamma (\nu_\beta|\mathcal{H}^0_q|g_\gamma)(\underline{\underline{\chi}}^{-1})_{\gamma\alpha} + \sum_\gamma (\nu_\beta|\mathcal{H}^1_q|g_\gamma)(\underline{\underline{\chi}}^{-1})_{\gamma\alpha} \\ \Gamma_{\beta\alpha}(z) &= \sum_\gamma (\nu_\beta|\mathcal{H}^1_q\mathcal{Q}\frac{1}{z-\mathcal{Q}(\mathcal{H}^0_q+\mathcal{H}^1_q)\mathcal{Q}}\mathcal{Q}\mathcal{H}^1_q|g_\gamma)(\underline{\underline{\chi}}^{-1})_{\gamma\alpha} \quad . \end{aligned} \tag{15}$$

In the lowest nonvanishing order the perturbation in the denominator of the memory kernel can be neglected and one obtains with the help of eq.(14) the second order result

$$\Gamma^{(2)}_{\beta\alpha}(z) = \sum_\gamma (\nu_\beta|\mathcal{H}^1_q\mathcal{Q}\frac{1}{z-\mathcal{H}^0_q}\mathcal{Q}\mathcal{H}^1_q|g_\gamma)(\underline{\underline{\chi}}^{-1})_{\gamma\alpha} \quad . \tag{16}$$

Inspecting the expression (16) on recognizes that it contains the projected resolvent $\mathcal{Q}(z - \mathcal{H}^0_q)^{-1}\mathcal{Q}$ instead of the full resolvent of zero order $(z - \mathcal{H}^0_q)^{-1}$ which governs the ordinary perturbation expansion mentioned at the beginning. This difference is crucial and the reason that the expression (16) is valid beyond the Schrödinger perturbation theory provided that the projection operator is chosen appropriately. To make this statement explicit let me mention that z takes values in the vicinity of the largest eigenvalue. If $\mathcal{H}^0_q$ allows for eigenvalues which become arbitrarily close to the largest eigenvalue as the parameter q varies the expression $(z - \mathcal{H}^0_q)^{-1}$ becomes singular even on a subspace which does not contain the eigenvector corresponding to the largest eigenvalue. As a consequence the radius of convergence of the Schrödinger perturbation expansion tends to zero. On the other hand the expression $\mathcal{Q}(z-\mathcal{H}^0_q)^{-1}\mathcal{Q}$ remains regular if the eigenelements corresponding to the resonant eigenvalues are incorporated in the projection operator (4). Therefore the expansion presented above is uniformly valid in q and the expression (16) is of second order in the perturbation. Hence one can hope that the frequency matrix determines the main behaviour according to eq.(7).

5 Discussion of an Example

For the discussion of the approximation schemes presented in the two preceding sections a simple exactly solvable model system that cointains nontrivial features should be investigated in this section from the viewpoint of the projection operator formalism. At the beginning of this section let us briefly review the properties of the model system. For a detailed discussion we refer to [13].

Consider the following one dimensional map

$$T_\epsilon(x) = \begin{cases} \gamma x, & x \in [-a,a] \\ -\gamma(x-2a), & x \in (a,\infty) \\ -\gamma(x+2a), & x \in (-\infty,a) \end{cases} \quad , \qquad \gamma := 2+\epsilon \tag{17}$$

which undergoes a symmetry breaking chaos transition in the limit $\epsilon \downarrow 0$. The characteristic exponent corresponding to the function

$$u(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{18}$$

shows a phase transition at $q = 0$ in the limit $\epsilon \downarrow 0$. This transition is governed by the degeneracy of the two largest eigenvalues $\exp(\pm q)$ of the transfer operator (2) At the bifurcation point $\epsilon = 0$ the map $T_{\epsilon=0}$ admits a simple Markov partition consisting of 6 intervals.

$$\begin{array}{ll} I_0^+ := [0,a] & I_0^- := [-a,0] \\ I_1^+ := [a,2a] & I_1^- := [-2a,-a] \\ I_2^+ := [2a,3a] & I_2^- := [-3a,-2a] \end{array} \tag{19}$$

On this partition the matrix representation of the transfer operator reads

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \exp(-q) & \exp(-q) & 0 & 0 & \exp(q) \\ 0 & \exp(-q) & \exp(-q) & 0 & 0 & \exp(q) \\ \exp(-q) & 0 & 0 & \exp(q) & \exp(q) & 0 \\ \exp(-q) & 0 & 0 & \exp(q) & \exp(q) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{20}$$

The right and left eigenvectors corresponding to the eigenvalues $\exp(\pm q)$ yield the following eigenfunctions and eigenmeasures of the transfer operator and its adjoint

$$\begin{aligned} \lambda_q^{(1)} &= \exp(q) \\ h^{(1)}(x) &= \chi_{I_0^+ \cup I_1^+}(x) \\ d\nu^{(1)} &= \nu^{(1)}(x)dx, \quad \nu^{(1)}(x) = \frac{1}{2a}\left(\chi_{I_0^+ \cup I_1^+}(x) + \exp(-2q)\chi_{I_2^-}(x)\right) \\ \lambda_q^{(2)} &= \exp(-q) \\ h^{(2)}(x) &= \chi_{I_0^- \cup I_1^-}(x) \\ d\nu^{(2)} &= \nu^{(2)}(x)dx, \quad \nu^{(2)}(x) = \frac{1}{2a}\left(\chi_{I_0^- \cup I_1^-}(x) + \exp(2q)\chi_{I_2^+}(x)\right) \end{aligned} \tag{21}$$

where $\chi_I(x)$ denotes the characteristic function of the interval I. The characteristic exponent (1) shows a typical scaling behaviour in the vicinity of the bifurcation point. Due to this fact the investigation of the parameter region $0 < \epsilon \ll 1$ yields a suitable test for the approximation schemes presented above.

Continued fraction expansion: For writing down the expansion (13) the matrix elements (11) have to be computed. We will see that a restriction to second order terms suffices. From eqs.(11) and (10) one gets the following relations

$$\begin{aligned}
\Omega_{11} &= H_1 \\
\Omega_{21} &= H_2 - H_1^2 \\
\Omega_{22}\Omega_{21} &= H_3 - 2H_1H_2 + H_1^3 \\
\Omega_{32}\Omega_{21}^2 &= (H_4 - H_2^2)(H_2 - H_1^2) - (H_3 - H_1H_2)^2
\end{aligned} \tag{22}$$

where

$$H_i := (\nu_*|(\mathcal{H}_q^u)^i|h_*) \tag{23}$$

denote the moments of the transfer operator with respect to the SRB measure. For the special model system under consideration these expressions can be computed easily even without having explicit knowledge of the invariant distribution (Appendix C). Finally one gets for the matrix elements (22)

$$\begin{aligned}
\Omega_{11} &= \cosh(q) \\
\Omega_{21} &= (1-4\delta)\sinh^2(q) \\
\Omega_{22} &= \frac{1-8\delta}{1-4\delta}\cosh(q) \\
\Omega_{32} &= -\frac{-16\delta^2}{(1-4\delta)^2} - \frac{32\delta^2(1-2\delta)}{(1-4\delta)^2}\sinh^2(q) \quad .
\end{aligned} \tag{24}$$

Here δ denotes the SRB measure of the interval $(2a, \infty)$. It has the meaning of the transiton rate from the region $x > 0$ to the region $x < 0$ and is a small quantity of order ϵ (cf. Appendix C). As the matrix element (24_4) is an order of magnitude smaller than the other ones a truncation of the continued fraction after the second term is justified in the limit of small ϵ. Then the resolvent reads

$$(\nu_1|\frac{1}{z-\mathcal{H}_q^u}|g_1) \simeq \cfrac{1}{z-\Omega_{11}-\cfrac{\Omega_{21}}{z-\Omega_{22}}} \tag{25}$$

which was called a two pole approximation in [10]. The singularities of this expression which determine the eigenvalues of the transfer operator can be calculated easily and one gets the correct scaling behaviour for the eigenvalues in the vicinity of the phase transition point (cf. [13]).

$$q =: \frac{4\delta}{2}\kappa, \quad \ln\lambda_q^{(1/2)} =: \frac{4\delta}{2}\psi_q^{(1/2)}, \qquad \psi_q^{(1/2)} \simeq -1 \pm \sqrt{1+\kappa^2} \tag{26}$$

Perturbation expansion: For the perturbative treatment we take the transfer operator corresponding to the map $T_{\epsilon=0}$ as the unperturbed part. Refering back to section 4 we build up the relevant sets with the eigenelements (21) which belong to a part of the spectrum of $\mathcal{H}_q^0$ that is well separated from the remaining one [13]. With eqs.(15) and (21) a simple straightforward calculation yields the frequency matrix

$$\begin{aligned}
\Omega_{11} &= \int\int \nu^{(1)}(x)\delta(x-T_\epsilon(y))h^{(1)}(y)dydx = \frac{2}{\gamma}\exp(q) \\
\Omega_{12} &= \int\int \nu^{(1)}(x)\delta(x-T_\epsilon(y))h^{(2)}(y)dydx = \frac{\epsilon}{\gamma}\exp(-3q) \\
\Omega_{21} &= \int\int \nu^{(2)}(x)\delta(x-T_\epsilon(y))h^{(1)}(y)dydx = \frac{\epsilon}{\gamma}\exp(3q) \\
\Omega_{22} &= \int\int \nu^{(2)}(x)\delta(x-T_\epsilon(y))h^{(2)}(y)dydx = \frac{2}{\gamma}\exp(-q)
\end{aligned} \tag{27}$$

Refering to the discussion made in section 4 it seems reasonable that the memory matrix is regular and of second order in the vicinity of the phase transition point $q=0, z=1$. It is indeed possible to derive the following estimate

$$\underline{\underline{\Gamma}}^{(2)}(z) = O(\epsilon^{2-\eta}), \quad \text{if } |z| > \exp(|q| - \eta\ln 2), \quad \eta > 0 \text{ arbitrary} \quad . \tag{28}$$

We devote appendix D to the tedious computation. Inspecting eq.(7) the poles of the resolvent are determined by the eigenvalues of the frequency matrix

$$\underline{\underline{\Omega}}^{(0)} + \underline{\underline{\Omega}}^{(1)} = \begin{pmatrix} (1-\frac{\epsilon}{2})\exp(q) & \frac{\epsilon}{2}\exp(-3q) \\ \frac{\epsilon}{2}\exp(3q) & (1-\frac{\epsilon}{2})\exp(-q) \end{pmatrix} \tag{29}$$

up to the lowest nonvanishing order in the small parameter ϵ. Introducing again the abbreviations $q = \kappa\epsilon/2$ and $\ln\lambda_q = \psi_q\epsilon/2$ one recovers the scaling behaviour (26).

6 Summary

Based on the projection operator approach used in statiscical mechanics the resolvent of the transfer operator has been investigated. The computation of the characteristic exponent has been reduced to the determination of the frequency and memory matrix (8). For the construction of a projection operator we have refered back to a dual space structure which is naturally connected to the dynamics generated by T_ϵ. Two approximation schemes have been proposed which on one hand allow for an explicit evaluation of these quantities and on the other hand define the relevant functions and measures which determine the projection operator. The first approach starts from the minimal set containing the SRB measure and extends it in a systematic way. By this procedure a continued fraction expansion of the resolvent has been obtained which involves the moments of the transfer operator. The second one is based on a perturbative approach starting from an unperturbed part of the transfer operator. The relevant sets are determined by the upper part of the spectrum of this operator which should be separated from the remainder. Also in this case only simple matrix elements of the transfer operator have to be evaluated. Both schemes have been applied to a simple model system exhibiting a symmetry breaking bifurcation. The correct behaviour in the vicinity of the phase transition point was reproduced. Additionally both approaches lead to a systematic expansion in the relevant small parameter. From these results we draw the conclusion that the proposed formalism may be helpful for a perturbative treatment of more complicated dynamical systems.

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Appendix A

From the definition (10) we get the orthogonality relations

$$\chi_{ij} = (\nu_i|g_j) = \delta_{ij}(\nu_i|g_i) \quad . \tag{30}$$

From the same equation we obtain that $\mathcal{H}_q^u g_i$ $({\mathcal{H}_q^u}^\dagger \nu_i)$ can be represented by a sum of the elemets $\{g_1, \ldots, g_{i+1}\}$ $(\{\nu_1, \ldots, \nu_{i+1}\})$. Refering to the definition (8_1) and to eq.(30) the non vanishing elements of the frequency matrix read

$$\Omega_{i-1\,i} = \frac{(\nu_{i-1}|\mathcal{H}_q^u|g_i)}{(\nu_i|g_i)}, \quad \Omega_{i\,i} = \frac{(\nu_i|\mathcal{H}_q^u|g_i)}{(\nu_i|g_i)}, \quad \Omega_{i+1\,i} = \frac{(\nu_{i+1}|\mathcal{H}_q^u|g_i)}{(\nu_i|g_i)} \quad . \tag{31}$$

Multiplying eq.(10_3) with $(\nu_{i+1}|$ and eq.(10_4) with $|g_{i+1})$ we obtain with the help of the orthogonality relation (30)

$$\begin{aligned} (\nu_{i+1}|g_{i+1}) &= (\nu_{i+1}|\mathcal{H}_q^u|g_i) \\ (\nu_{i+1}|g_{i+1}) &= ({\mathcal{H}_q^u}^\dagger \nu_i|g_{i+1}) \quad . \end{aligned} \tag{32}$$

Combining eqs.(31) and (32) yields eq.(11).

From the remark made at the beginning of the Appendix we get immediately

$$\mathcal{Q}\mathcal{H}_q^u g_i = 0, \quad \mathcal{Q}^\dagger {\mathcal{H}_q^u}^\dagger \nu_i = 0, \quad i = 1, \ldots, N-1 \quad . \tag{33}$$

From this relation the result (12) is obvious.

Appendix B

To show the identity to the continued fraction expansion proposed in [10] we introduce the resolvent matrix

$$R_{\beta\alpha}(z) := (\nu_\beta|(z - \mathcal{H}_q^u)^{-1}|g_\alpha) \quad . \tag{34}$$

Then eq.(7) yields

$$\delta_{i1}(\nu_1|g_1) = \Big((z\underline{\underline{1}} - \underline{\underline{\Omega}} - \underline{\underline{\Gamma}}(z))\underline{\underline{R}}(z)\Big)_{i1} \quad . \tag{35}$$

Using the expressions (11), (12) and the abbreviation

$$\Psi_i(z) := \frac{R_{i+1\,1}(z)}{R_{i\,1}(z)} \tag{36}$$

eq.(35) reads

$$\begin{aligned} (\nu_1|g_1) &= (z-\Omega_{11}-\Psi_1(z))R_{11}(z) \\ 0 &= -\Omega_{i\,i-1}+(z-\Omega_{ii}-\Psi_i(z))\Psi_{i-1}, \quad i=2,...,N-1 \\ 0 &= -\Omega_{N\,N-1}+(z-\Omega_{NN}-\Gamma_{NN}(z))\Psi_{N-1} \quad . \end{aligned} \tag{37}$$

To get the formal equivalence one has to introduce the notation

$$\mu := \frac{1}{z}, \quad [\mu]_0 := zR_{11}(z), \quad [\mu]_i := z\Psi_i(z), \quad i=1,\ldots,N-1 \quad . \tag{38}$$

Then eq.($37_{1,2}$) ends up in

$$\begin{aligned} (\nu_1|g_1) &= (1-\mu\Omega_{11}+\mu^2[\mu]_1)[\mu]_0 \\ \Omega_{i\,i-1} &= (1-\mu\Omega_{ii}-\mu^2[\mu]_i)[\mu]_{i-1} \end{aligned} \tag{39}$$

which is identical to eq.(10) of ref.[10].

The formal continued fraction expansion stops if at some stage the expression $(\nu_i|g_i)$ vanishes. Let us therefore assume that the construction of section 3 holds up to the order N and denote the relevant sets by $\{g_1,\ldots,g_N\}$ and $\{\nu_1,\ldots,\nu_N\}$. Furthermore the construction should fail at the next level which means

$$g_c := \prod_{k=1}^{N} \mathcal{Q}_k\mathcal{H}_q^u g_N, \quad \nu_c := \prod_{k=1}^{N} \mathcal{Q}_k^\dagger {\mathcal{H}_q^u}^\dagger \nu_N, \quad (\nu_c|g_c)=0 \quad . \tag{40}$$

There is no unique way to circumvent this difficulty. One possible way is to omit the expressions (40) from the relevant set and incorporate in the next step the following function and measure

$$g_{N+1} := \prod_{k=1}^{N} \mathcal{Q}_k\mathcal{H}_q^u g_c, \quad \nu_{N+1} := \prod_{k=1}^{N} \mathcal{Q}_k^\dagger {\mathcal{H}_q^u}^\dagger \nu_c \tag{41}$$

provided $(\nu_{N+1}|g_{N+1}) \neq 0$. Also with these sets it is possible to write down a continued fraction expansion and extend it to higher orders. But as the relation (33) ceases to be valid for $i=N$ this expansion does not share the simple structure of eq.(13). We therefore omit the explicit expressions.

Appendix C

Consider the sets

$$\begin{aligned} J_{+1} := \{x|x>0\} \quad &, \quad J_{-1} := \{x|x<0\} \\ J_{i_0,\ldots,i_{k-1}} &:= \{x|T_\epsilon^j(x)\in J_{i_j},\, 0\le j\le k-1\} \quad . \end{aligned} \tag{42}$$

Then the moments (23) of the transfer operator read

$$
\begin{aligned}
H_k &= \int \exp\left(q\sum_{j=0}^{k-1} u(T_\epsilon^j(x))\right) h_*(x)dx \\
&= \sum_{i_0,\ldots,i_{k-1}} \exp\left(q\sum_{j=0}^{k-1} i_j\right) \int_{J_{i_0,\ldots,i_{k-1}}} h_*(x)dx \\
&= \sum_{i_0,\ldots,i_{k-1}} \exp\left(q\sum_{j=0}^{k-1} i_j\right) \mu_{i_0,\ldots,i_{k-1}} \quad .
\end{aligned} \tag{43}
$$

Here the fact that ν_* coincides with the Lebesgue measure and the abbreviation $\mu_{i_0,\ldots,i_{k-1}}$ for the SRB measure of the sets (42_2) has been used. To compute the last mentioned quantity we note that

$$
\begin{aligned}
\bigcup_{i_k} J_{i_0,\ldots,i_k} &= J_{i_0,\ldots,i_{k-1}} \\
\bigcup_{i_0} J_{i_0,\ldots,i_k} &= T_\epsilon^{-1}(J_{i_1,\ldots,i_k}), \quad k \geq 1
\end{aligned} \tag{44}
$$

so that the T_ϵ invariance of the SRB measure leads to

$$
\begin{aligned}
\sum_{i_k} \mu_{i_0,\ldots,i_k} &= \mu_{i_0,\ldots,i_{k-1}} \\
\sum_{i_0} \mu_{i_0,\ldots,i_k} &= \mu_{i_1,\ldots,i_k}, \quad k \geq 1 \quad .
\end{aligned} \tag{45}
$$

By the symmetry of the map we have

$$
\mu_{+1} = \mu_{-1} = \frac{1}{2} \quad . \tag{46}
$$

If ϵ is sufficiently small every trajectory leaving the interval J_{+1} (J_{-1}) stays in the other interval J_{-1} (J_{+1}) for a sufficiently long time. This means

$$
\begin{aligned}
0 &= \mu_{+1,-1,+1} = \mu_{-1,+1,-1} \\
0 &= \mu_{+1,-1,+1,+1} = \mu_{-1,+1,-1,-1} \\
0 &= \mu_{+1,+1,-1,+1} = \mu_{-1,-1,+1,-1} \\
0 &= \mu_{-1,+1,+1,-1} = \mu_{+1,-1,-1,+1} \quad .
\end{aligned} \tag{47}
$$

The remaining quantities can be determined uniquely from eqs.(45) in the case $k = 1, 2, 3$

$$
\begin{aligned}
\delta &:= \mu_{+1,-1} = \mu_{-1,+1} \\
\frac{1}{2} - \delta &= \mu_{+1,+1} = \mu_{-1,-1} \\
\delta &= \mu_{+1,-1,-1} = \mu_{-1,+1,+1} = \mu_{+1,+1,-1} = \mu_{-1,-1,+1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}-2\delta &= \mu_{+1,+1,+1}=\mu_{-1,-1,-1} \\
\delta &= \mu_{+1,+1,-1,-1}=\mu_{-1,-1,+1,+1} \\
\delta &= \mu_{+1,+1,+1,-1}=\mu_{-1,-1,-1,+1} \\
\delta &= \mu_{+1,-1,-1,-1}=\mu_{-1,+1,+1,+1} \\
\frac{1}{2}-3\delta &= \mu_{+1,+1,+1,+1}=\mu_{-1,-1,-1,-1} \quad .
\end{aligned}
\tag{48}
$$

Here the quantity defined in eq.(48_1) measures those points which leave one of the intervals (42_1) and can be identified with the escape rate. Refering to the tent like shape of the mapping (17) its value can be estimated to $\delta \simeq \epsilon a/(4a) = \epsilon/4$ [5]. Inserting eqs.(46-48) into eq.(43) we obtain

$$
\begin{aligned}
H_1 &= \cosh(q) \\
H_2 &= \cosh(2q)+2\delta(1-\cosh(2q)) \\
H_3 &= \cosh(3q)+4\delta(\cosh(q)-\cosh(3q)) \\
H_4 &= \cosh(4q)+2\delta(-3\cosh(4q)+2\cosh(2q)+1) \quad .
\end{aligned}
\tag{49}
$$

With eq.(22) one gets the result (24).

Appendix D

The matrix element $\Gamma_{11}(z)$ will be evaluated explicitly. The other elements can be calculated in the same way.

From the definition (2) and (21_2) we obtain

$$
(\mathcal{H}_q^u h^{(1)})(x) = \frac{2}{\gamma} e^q (\chi_{I_0^+ \cup I_1^+}(x) + \chi_{[2a,\gamma a]}(x))
\tag{50}
$$

so that by the definition (4) of the projection operator the relation

$$
(\mathcal{Q}\mathcal{H}_q^1 h^{(1)})(x) = (\mathcal{Q}\mathcal{H}_q^u h^{(1)})(x) = \frac{2}{\gamma} e^q \mathcal{Q}\chi_{[2a,\gamma a]}(x)
\tag{51}
$$

holds.

If ν is a measure with density $\nu(x)$ then by the definition of the adjoint operator ${\mathcal{H}_q^u}^\dagger$ the measure ${\mathcal{H}_q^u}^\dagger \nu$ is determined by the density $\exp(qu(x))\nu(T_\epsilon(x))$. This result follows by a straightforward computation. Using this relation one gets

$$
({\mathcal{H}_q^u}^\dagger \nu^{(1)})(x) = \frac{e^q}{2a}\left(\nu^{(1)}(x) - \chi_{K^+}(x)\right) + \frac{e^{-3q}}{2a}\chi_{K^-}(x)
\tag{52}
$$

where

$$
K^+ := [\frac{\gamma}{2a}, 2a - \frac{\gamma}{2a}], \qquad K^- := [-2a + \frac{\gamma}{2a}, -\frac{\gamma}{2a}]
\tag{53}
$$

[5] The exact expression reads $\delta = \epsilon/(2\gamma)$.

denote those parts of the intervals $I_0^+ \cup I_1^+$ and $I_0^- \cup I_1^-$ that are maped by T_ϵ on the intervals I_2^+ and I_2^-. From eq.(52) we obtain immediately

$$(\mathcal{Q}^\dagger \mathcal{H}_q^{1\,\dagger} \nu^{(1)})(x) = \frac{1}{2a} \mathcal{Q}^\dagger \left(e^{-3q} \chi_{K^-}(x) - e^q \chi_{K^+}(x) \right) \quad . \tag{54}$$

With eq.(51) and (54) eq.(16) yields for the element under consideration

$$\Gamma_{11}(z) = \frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} \int \frac{1}{2a} \left(e^{-3q} \chi_{K^-}(x) - e^q \chi_{K^+}(x) \right) \mathcal{Q} (\mathcal{H}_q^0)^k \frac{2}{\gamma} e^q \chi_{[2a,\gamma a]}(x) dx \quad . \tag{55}$$

For the following considerations we will assume that for a sufficiently large natural number N the relation $\epsilon = 2^{-N+1}$ holds. This assumption restricts the ϵ values to a countable sequence. But as this sequence tends to 0 it is reasonable to assume that it contains the main behaviour for small ϵ values. We then obtain

$$\begin{aligned} ((\mathcal{H}_q^0)^k \chi_{[2a,\gamma a]})(x) &= \frac{e^q}{2} \left(\frac{e^{-q}}{2} \right)^{k-1} \chi_{[-2^k \epsilon a, 0]}(x), \quad 1 \le k \le N \\ ((\mathcal{H}_q^0)^k \chi_{[2a,\gamma a]})(x) &\sim \chi_{I_0^- \cup I_1^-}(x), \quad k > N \end{aligned} \tag{56}$$

where the first relation follows by induction and the second from the requirement made above. With these relations the integrand in eq.(55) can be evaluated. One recognizes that in the case $k > N$ the summands are eliminated by the projection operator. For the remaining part we get by using the definition (4)

$$\begin{aligned} \Gamma_{11}(z) &= \frac{e^q}{z\gamma a} \int \left(e^{-3q} \chi_{K^-}(x) - e^q \chi_{K^+}(x) \right) \mathcal{Q} \chi_{[2a,\gamma a]}(x) dx \\ &\quad + \frac{e^q}{z\gamma a} \sum_{k=1}^{N-1} e^{2q} \left(\frac{e^{-q}}{2z} \right)^k \int \left(e^{-3q} \chi_{K^-}(x) - e^q \chi_{K^+}(x) \right) \mathcal{Q} \chi_{[-2^k \epsilon a, 0]}(x) dx \\ &= \frac{e^q}{z\gamma a} \left(- \frac{\epsilon^2 a e^{-q}}{\gamma} \right) + \frac{e^q}{z\gamma a} \sum_{k=1}^{N-2} e^{2q} \left(\frac{e^{-q}}{2z} \right)^k \left(- \frac{\epsilon^2 a 2^k e^{-3q}}{\gamma} \right) \quad . \end{aligned} \tag{57}$$

Finally we obtain

$$\Gamma_{11}(z) = - \frac{\epsilon^2}{\gamma^2} \frac{1}{z} \sum_{k=0}^{N-2} \left(\frac{e^{-q}}{z} \right)^k, \quad \epsilon = 2^{-N+1} \quad . \tag{58}$$

The other matrix elements can be calculated in a similar fashion leading to the result

$$\begin{aligned} \Gamma_{12}(z) &= \frac{\epsilon^2}{\gamma^2} \frac{e^{-2q}}{z} \sum_{k=0}^{N-2} \left(\frac{e^q}{z} \right)^k \\ \Gamma_{21}(z) &= \frac{\epsilon^2}{\gamma^2} \frac{e^{2q}}{z} \sum_{k=0}^{N-2} \left(\frac{e^{-q}}{z} \right)^k \\ \Gamma_{22}(z) &= - \frac{\epsilon^2}{\gamma^2} \frac{1}{z} \sum_{k=0}^{N-2} \left(\frac{e^q}{z} \right)^k \quad . \end{aligned} \tag{59}$$

Using the estimate

$$
\begin{aligned}
\left| \epsilon^2 \sum_{k=0}^{N-2} \left(\frac{e^{\pm q}}{z} \right)^k \right| &= \epsilon^{2-\eta} \left| \sum_{k=0}^{N-2} \left(\frac{e^{\pm q}}{z} \right)^k 2^{-\eta(N-1)} \right| \\
&\leq \epsilon^{2-\eta} \sum_{k=0}^{\infty} \left| \frac{e^{\pm q}}{2^\eta z} \right|^k \qquad (60)
\end{aligned}
$$

one recognizes that the matrix elements (58) and (59) are of order $O(\epsilon^{2-\eta})$ if the series on the right hand side of eq.(60) converges.

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