Equilibrium Phase Transitions in Coupled Map Lattices: A Pedestrian Approach

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Abstract

A class of piecewise linear coupled map lattices with simple symbolic dynamics is constructed. It can be solved analytically in terms of the statistical mechanics of spin lattices. The corresponding Hamiltonian is written down explicitly in terms of the parameters of the map. The approach follows the line of recent mathematical investigations. But the presentation is kept elementary so that phase transitions in the dynamical model can be studied in detail. Although the method works only for map lattices with repelling invariant sets some of the conclusions, i.e. the role of local curvature of the single site map and properties of the nearest neighbour coupling might play an important role for phase transitions in general dynamical systems.

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1 Introduction

Coupled map lattices have been introduced more than a decade ago as a paradigm for studying general features of coherent pattern formation in spatially extended systems

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[10]. Since these models contain basic features, i.e. the competition between local chaos and spatial interaction in a very efficient way most investigations have been based on numerical analysis. One of the prominent properties of these systems are qualitative changes of the dynamical behaviour which are intimately related to the limit of large system size. These phenomena are usually called phase transitions in contrast to plain bifurcations for which the spatial extension plays a minor role. In fact, numerical simulations on particular piecewise linear antisymmetric maps display a phase transition like behaviour when placed on a two–dimensional square lattice [12]. Although these features resemble Ising phase transitions a detailed analysis shows slight deviations from Ising universality [11]. It is quite unclear whether hidden long range correlations are responsible for these deviations or whether equilibrium like behaviour is restored in the dynamical model on larger length scales [5]. Hence, phase transitions in spatially extended dynamical systems are to date not very well understood from the general point of view.

Of course, one has to clarify why general dynamical models have something in common with equilibrium statistical mechanics at all. Such a link rests on a quite old idea which dates back at least several decades. One uses to some extent the symbolic dynamics of a chaotic system in order to establish an equivalence between its ergodic properties and the statistical mechanics of spin systems (cf. [1] for an elementary introduction). Such approaches have been used to rigorously relate weakly coupled fully chaotic map lattices with the high temperature phase of spin lattices [4, 3, 6]. Hence, space–time chaos can be understood as a state where all correlations decay exponentially. It is tempting to look for whether the appearance of coherent patterns from space–time chaotic states is related to equilibrium phase transitions in the corresponding symbolic dynamics. General suggestions can be found in different contributions (cf. e.g. [8]) but without making the argument explicit. For particular cases like globally coupled models [9] or discontinuous maps [7] such investigations have been performed recently. Thus, there is striking evidence that coherent pattern formation has much in common with equilibrium phase transitions.

Here I am going to highlight these concepts by using an elementary approach and avoiding the sometimes heavy mathematical notation. From the principle point of view one may claim that the content is not completely new. However, by concentrating on simple piecewise linear maps the main features will be illustrated which allow to construct the equivalence between dynamical systems and Ising spin lattices. In particular, the interactions of the corresponding spin Hamiltonian are expressed in terms of parameters of the dynamical model. Therefore, it is particularly easy to detect phase transitions. But we have to pay a price for the simplicity of the underlying construction. The approach works only for repellers. Such a shortcoming might look slightly odd from a superficial physical point of view, but it plays to some extent a minor role if one has dynamical systems theory in mind. It is in fact widely known that analytical approaches, in particular those based on symbolic dynamics are often technically much more difficult to apply for systems having attracting sets. This feature is already reflected by the celebrated Smale horseshoe and transfers to some extent to all hyperbolic maps of the plane. Thus to comply with what we have promised in the title, i.e. using a completely elementary approach one is forced to investigate systems with repelling invariant sets.

2 The Ising map

To begin with let me introduce the single site map. Consider the piecewise linear map depicted in figure 1, $f : I_- \cup I_+ \to I$, where $I = [a_{--}, a_{+-}]$, $I_- = [a_{--}, a_{-+}]$, $I_+ = [a_{++}, a_{+-}]$ for some $a_{--} < a_{-+} < a_{++} < a_{+-}$, and endpoints of the intervals $I_\pm$ are mapped to endpoints. If $|f'(x)| > 1$, the so called cylinder sets

$$I_{\sigma_0,\ldots,\sigma_n} := \{x \in I \,|\, f^k(x) \in I_{\sigma_k},\, 0 \leq k \leq n\} \tag{1}$$

shrink exponentially in size and lead to a binary symbolic dynamics in the obvious way, $f^k(x) \in I_{\sigma_k}$, $\sigma_k \in \{-,+\}$. Since the map is not defined on the whole interval I the invariant set is a well known type of Cantor repeller.

So far the topological features are completely fixed. In order to discuss ergodic properties one needs to specify an invariant measure. First of all, calculating invariant measures is usually a difficult task if the slope of the map is not constant. Let me therefore restrict to the case that the map has constant slope on the cylinder sets of the second generation

$$\gamma_{\tilde{\sigma}\sigma} := |f'(x)|_{x \in I_{\tilde{\sigma},\sigma}} \quad . \tag{2}$$

Thus for $\gamma_{\tilde{\sigma}+} \neq \gamma_{\tilde{\sigma}-}$ the slope of the map is not constant on the sets $I_{\tilde{\sigma}}$ of the partition and reflects in a very elementary way a finite amount of curvature of the map. Furthermore, there does not seem to be a natural invariant measure available since we are dealing with repelling sets. Lebesgue typical initial conditions will finally leave the domain of interest. However, there exists a generalisation of a natural invariant measure for repelling sets which just takes the local slopes into account. Formally this measure maximises the topological pressure corresponding to the local expansion rate a property which is shared by SRB measures in systems with attracting sets. The measure assigns a probability to the cylinder sets (1) according to the expression

$$\mu(I_{\sigma_0,\ldots,\sigma_n}) = h_{\sigma_0\sigma_1} \frac{\nu_{\sigma_{n-1}\sigma_n}}{\lambda^{n+1}} \prod_{k=0}^{n-2} \frac{1}{\gamma_{\sigma_k\sigma_{k+1}}} \quad . \tag{3}$$

Eq.(3) can be either calculated from the condition of invariance or by employing transfer operator techniques [13]. Here λ is the largest eigenvalue of the transfer

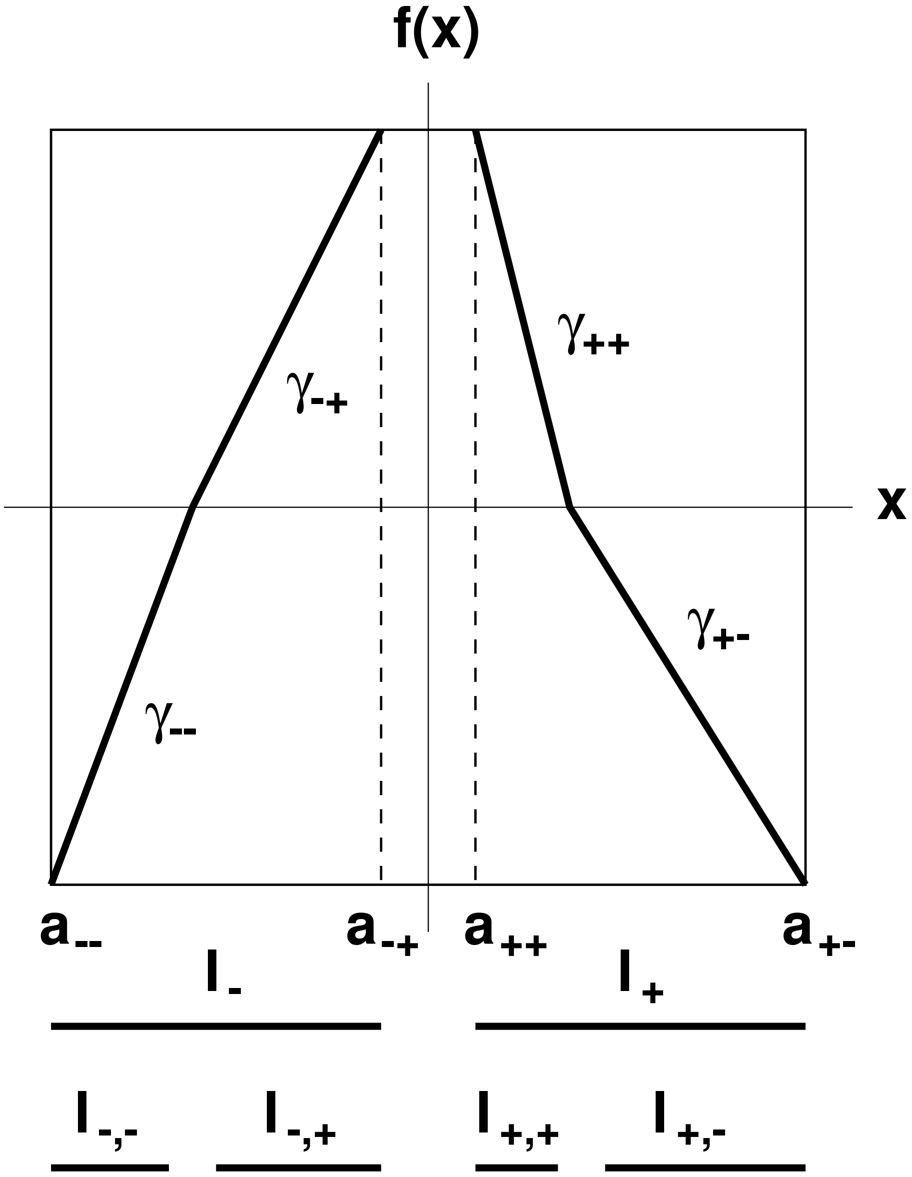

Figure 1: Piecewise linear single site map with invariant Cantor set. The first two generations of cylinder sets are indicated. $\gamma_{\tilde{\sigma}\sigma}$ denotes the modulus of the slope.

matrix

$$\underline{\underline{H}} = \begin{pmatrix} 1/\gamma_{--} & 0 & 0 & 1/\gamma_{+-} \\ 1/\gamma_{--} & 0 & 0 & 1/\gamma_{+-} \\ 0 & 1/\gamma_{-+} & 1/\gamma_{++} & 0 \\ 0 & 1/\gamma_{-+} & 1/\gamma_{++} & 0 \end{pmatrix} \tag{4}$$

and determines the escape rate of the repeller. $(h_{--}, h_{-+}, h_{++}, h_{+-})$ denotes the corresponding right– and $(\nu_{--}, \nu_{-+}, \nu_{++}, \nu_{+-})$ the corresponding left–eigenvector. Both are normalised according to $\underline{\nu} \cdot \underline{h} = 1$. The explicit expression for these quantities do not matter for our purpose. It is however important that the probabilities (3) are given in terms of the product of slopes which the trajectory encounters during its itinerary, if we discard boundary terms appearing for σ_0 and σ_n. These boundary terms would be absent if the measure were based on an expansion in terms of periodic orbits. Hence, the boundary contributions will not play an essential role in our discussion.

With the measure (3) mean values of quantities, i.e. integrals involving phase space functions can be calculated. Using a partition by cylinder sets the mean value of a phase space function g reads

$$\langle g \rangle = \int g \, d\mu = \lim_{n\to\infty} \sum_{\sigma_0,\ldots,\sigma_n} g(x_{\sigma_0,\ldots,\sigma_n}) \mu(I_{\sigma_0,\ldots,\sigma_n}) \quad , \tag{5}$$

where $x_{\sigma_0,\ldots,\sigma_n}$ denotes a point in the cylinder set under consideration. The particular choice of the point does not matter as long as the observable g is Hölder continuous (cf. e.g. [2]). The right hand side of eq.(5) has the form of a canonical thermodynamic expectation value for a spin chain, if we define the negative logarithm of the probability (3) as the Hamiltonian. If we discard the above mentioned boundary contributions the Hamiltonian follows from eq.(3) as

$$H_{\sigma_0,\ldots,\sigma_n} = \sum_{k=0}^{n-1} \ln \gamma_{\sigma_k \sigma_{k+1}} + (n+1) \ln \lambda \quad . \tag{6}$$

The spatial extension of the spin chain corresponds to the temporal evolution in the dynamical system. Eq.(6) yields the usual Ising chain Hamiltonian with nearest neighbour pair interaction. The exchange constant and the external field are determined by $\ln(\gamma_{--}/\gamma_{-+}) + \ln(\gamma_{++}/\gamma_{+-})$ and $\ln(\gamma_{++}/\gamma_{--})$ respectively. Whereas the exchange constant is given in terms of the curvature of the map the external field is essentially related to symmetry properties of the map. Finally, the escape rate of the repeller contributes to the ground state energy. Since the system is one–dimensional and has short range interaction no phase transition occurs, i.e. the mean values depend smoothly on the parameters of the system.

All the previous considerations are special cases of a much more general mathematical approach (cf. e.g. [14]). The important point which I want to stress here is that these advanced formalisms become simple for piecewise linear maps. The latter are defined to some extent in a geometric way. Fixing partitions $\{U_\sigma\}$ and $\{U_{\tilde{\sigma},\sigma}\}$ one defines the map $f : U_{\tilde{\sigma},\sigma} \to U_\sigma$ in such a way that it acts linear on each set. These considerations do not depend on the dimension of the space under consideration and we are going to carry them over for the construction of coupled map lattices and the associated statistical mechanics.

3 Coupled Cantor repellers

Let us consider a spatially one–dimensional map lattice consisting of L lattice sites where periodic boundary conditions are imposed. In order to study the limit of infinite system size we will consider the limit $L \to \infty$ after statistical quantities like the invariant measure have been determined. Phase transitions, i.e. the occurrence of different ergodic components can then be detected by including a small symmetry breaking field. Such an approach avoids the difficulties which are related to the implementation of an infinite dimensional system at the very beginning.

We will consider single site maps of the type introduced in the previous section (cf. figure 1) so that the whole phase space is given by the L–dimensional cube I^L. Coordinates of the single site map are denoted by $x^{(\nu)}$, $0 \le \nu \le L-1$. If we first consider the system without spatial coupling then the uncoupled map lattice T_0 maps each coordinate according to the single site map. Since each map carries its cylinder sets the cylinder sets of the uncoupled model are given by direct products, i.e.

$$U_{\underline{\sigma}} = I_{\sigma^{(0)}} \otimes \ldots \otimes I_{\sigma^{(L-1)}} \tag{7}$$

and

$$\begin{aligned} U_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}} &= \{\underline{x} \in I^L \,|\, T_0^k(\underline{x}) \in U_{\underline{\sigma}_k}, 0 \le k \le n-1\} \\ &= I_{\sigma_0^{(0)},\ldots,\sigma_{n-1}^{(0)}} \otimes \ldots \otimes I_{\sigma_0^{(L-1)},\ldots,\sigma_{n-1}^{(L-1)}} \end{aligned} \tag{8}$$

where $\underline{\sigma} = (\sigma^{(0)}, \ldots, \sigma^{(L-1)})$ enumerates the different elements of the partition (7). According to eq.(8) the symbolic dynamics is now given in terms of two–dimensional symbol lattices where one direction corresponds to the spatial dimension of the dynamical system and the other to temporal evolution. The uncoupled map lattice acts linear on the cylinder sets of the second generation

$$T_0 : U_{\underline{\tilde{\sigma}},\underline{\sigma}} \to U_{\underline{\sigma}} \tag{9}$$

and the determinant of the Jacobian is just given by the product of the local slopes

$$|\det(DT_0(\underline{x}))|_{\underline{x}\in U_{\underline{\tilde{\sigma}},\underline{\sigma}}} = \gamma_{\tilde{\sigma}^{(0)}\sigma^{(0)}} \ldots \gamma_{\tilde{\sigma}^{(L-1)}\sigma^{(L-1)}} \quad . \tag{10}$$

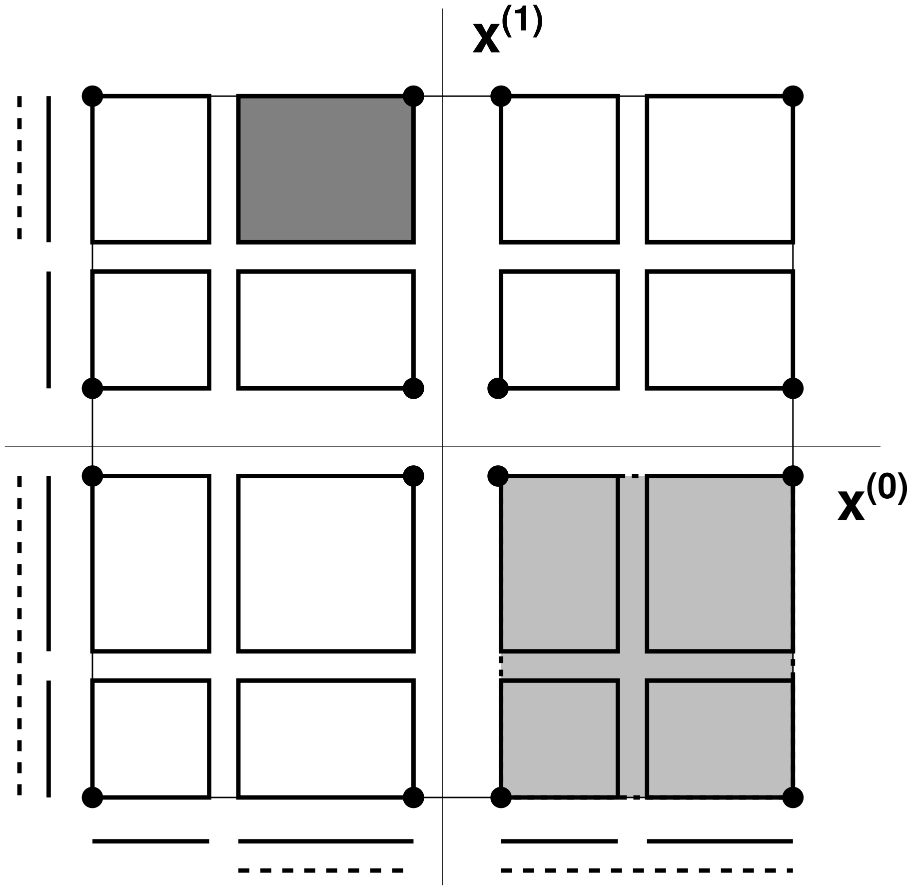

Figure 2: Diagrammatic view of cylinder sets of the second generation $U_{\underline{\tilde{\sigma}},\underline{\sigma}}$ for two maps without coupling (cf. figure 1). Greyshading indicates a particular preimage ($U_{-+,+-}$) and its image ($U_{+,-}$) with respect to the piecewise linear transformation T_0. Dots mark the vertices of sets of the first generation. Full lines indicate cylinder sets of the single site map and broken lines a particular preimage and its image.

For the case of two maps the whole phase space structure is depicted in figure 2. It is important to emphasise that the absence of spatial coupling is reflected by the product structure of the cylinder sets.

We are now going to introduce a spatial coupling in our model. Considering a coupling function $\Phi : I^L \to I^L$ let us introduce the coupled map lattice

$$T = T_0 \circ \Phi \quad . \tag{11}$$

If we just assume some analytical expression for Φ the simple phase space structure which has been just mentioned might be destroyed. In order to keep the whole analysis simple we use a different approach. We first introduce cylinder sets and then define the appropriate coupling function. As mentioned above a real spatial coupling has to remove the product structure from the cylinder sets. We therefore keep the sets of the first generation $\{U_{\underline{\sigma}}\}$ and deform the cubes of the second generation $\{V_{\underline{\tilde{\sigma}},\underline{\sigma}}\}$ so that the product structure is removed. Since the outer vertex of $U_{\underline{\tilde{\sigma}},\underline{\sigma}}$ is fixed by the vertex of $U_{\underline{\tilde{\sigma}}}$, the deformed cube $V_{\underline{\tilde{\sigma}},\underline{\sigma}}$ is uniquely determined by specifying all of its (one–dimensional) edges. The situation is depicted in figure 3 for the case of two maps. Finally we require that from the point of view of symbolic dynamics the coupled map lattice inherits its properties from the uncoupled one, i.e. it acts as (cf. eq.(9))

$$T : V_{\underline{\tilde{\sigma}},\underline{\sigma}} \to U_{\underline{\sigma}} \quad , \tag{12}$$

and we require that T is linear on each cube $V_{\underline{\tilde{\sigma}},\underline{\sigma}}$. Hence edges are mapped on edges. If we denote by $t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}}$ the ratio between the edges of cube $V_{\underline{\tilde{\sigma}},\underline{\sigma}}$ and of $U_{\underline{\sigma}}$ extending along the ν–axis, then the value of the determinant of the Jacobian reads

$$|\det(DT(\underline{x}))|_{\underline{x}\in V_{\underline{\tilde{\sigma}},\underline{\sigma}}} = \frac{\lambda(U_{\underline{\sigma}})}{\lambda(V_{\underline{\tilde{\sigma}},\underline{\sigma}})} = \prod_{\nu=0}^{L-1} t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}} \quad , \tag{13}$$

where $\lambda(.)$ denotes the volume of the corresponding cube. The just mentioned ratios $t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}}$ are the important parameters of our geometrically introduced map lattice since our considerations imply the analytical expression

$$(T(\underline{x}))^{(\nu)} = -\tilde{\sigma}^{(\nu)} t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}} \left(x^{(\nu)} - a_{\tilde{\sigma}^{(\nu)}\sigma^{(\nu)}} \right) + \sigma^{(\nu)}, \quad x \in V_{\underline{\tilde{\sigma}},\underline{\sigma}} \quad . \tag{14}$$

For the uncoupled case T_0 the local slopes read $t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}} = \gamma_{\tilde{\sigma}^{(\nu)}\sigma^{(\nu)}}$. Spatial interaction between the maps is caused by the dependence of the local slopes $t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}}$ on neighbouring lattice sites. In addition, the coupling function (cf. eq.(11)) is easily read off as

$$(\Phi(\underline{x}))^{(\nu)} = \frac{t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}}}{\gamma_{\tilde{\sigma}^{(\nu)}\sigma^{(\nu)}}} \left(x^{(\nu)} - a_{\tilde{\sigma}^{(\nu)}\sigma^{(\nu)}} \right) + a_{\tilde{\sigma}^{(\nu)}\sigma^{(\nu)}} \quad x \in V_{\underline{\tilde{\sigma}},\underline{\sigma}} \quad . \tag{15}$$

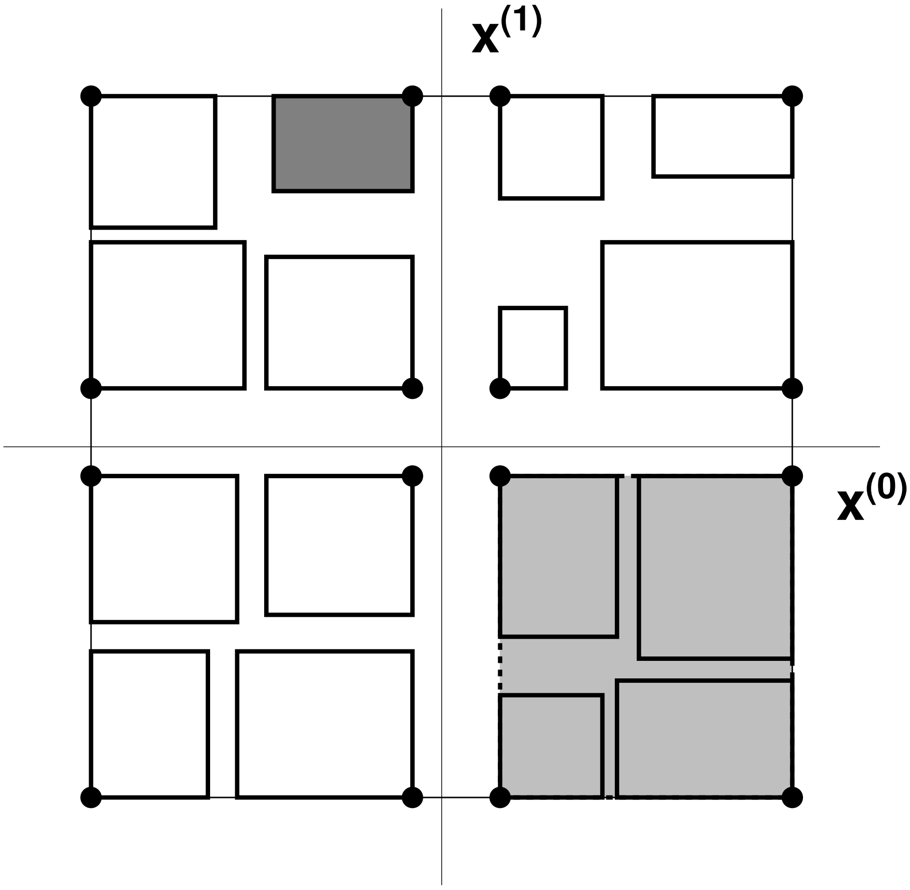

Figure 3: Diagrammatic view of cylinder sets of the second generation $V_{\underline{\tilde{\sigma},\sigma}}$ for two coupled maps (cf. figure 2). Greyshading indicates a particular preimage ($V_{-+,+-}$) and its image ($U_{+,-}$) with respect to the piecewise linear transformation T. Dots indicate the vertices of sets of the first generation. They coincide with the values of the uncoupled map.

These explicit analytical expressions for the map are however not necessary to determine the invariant measure which is completely specified in terms of the local expansion rate (13). Since the symbolic dynamics has not changed the measure is given by the product of these expansion rates, if boundary terms are discarded (cf. eq.(3)). Hence the corresponding spin Hamiltonian reads

$$H_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}} = -\ln \mu(U_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}}) = \sum_{k=0}^{n-1} \sum_{\nu=0}^{L-1} \ln t^{(\nu)}_{\underline{\sigma}_k,\underline{\sigma}_{k+1}} \tag{16}$$

where periodic boundary conditions are imposed for simplicity. The long time averages of the model (14) are now given by canonical averages with the two–dimensional spin Hamiltonian (16) in the thermodynamic limit of infinite system size . The local slopes of the map lattice translate into spin interactions. In particular the choice

$$t^{(\nu)}_{\underline{\tilde{\sigma}},\underline{\sigma}} = \exp\left[-h\tilde{\sigma}^{(\nu)} - J\tilde{\sigma}^{(\nu)}\left(\sigma^{(\nu)} + \tilde{\sigma}^{(\nu+1)}\right) + e_0\right] \tag{17}$$

yields the isotropic nearest neighbour coupled Ising model. Then, in the paramagnetic phase ($h=0$, $J < J_c = \ln(\sqrt{2}+1)/2$) the dynamics of eq.(14) is space–time mixing whereas for $h=0$, $J > J_c$ the 'inversion symmetry' of the dynamical model is spontaneously broken. Because of the construction of the coupled map lattice the local slopes are flipped according to the symbol at neighbouring lattice sites. Therefore the type of coupling of the coupled map lattice, e.g. a nearest neighbour coupling, translates directly to the corresponding Hamiltonian. This feature made the analysis particularly simple in contrast to the general case where the estimates for the spin interaction introduce considerable technical difficulties into the analysis [3]. In our case, the phase transition is caused by the modulation of the local slopes, where for the choice (17) a change of slope by a factor of 3 is already sufficient to induce the phase transition. Finally, one should mention that the map lattice (14) is easily extended to a continuous function on each set $U_{\underline{\sigma}}$ using homotopies.

4 Conclusion

By means of a geometrical construction we have introduced a class of piecewise linear coupled map lattices for which for the corresponding statistical mechanics in terms of two–dimensional spin lattices can be supplied easily. Dynamical models can be realised where Ising–like phase transitions appear. To trigger these phase transitions two ingredients are necessary. Apart from curvature of the single site map a spatial coupling was imposed which modulates the local slopes of the single site dynamics. Whether such features are typical for realising phase transitions in expanding map lattices is of course an open question and deserves further investigations. At least, it is possible to test this conjecture numerically in map lattices having attracting invariant sets.

Most systems for which a statistical mechanics has been explicitly constructed has lead to quite complicated and highly anisotropic spin models [3, 9]. Thus our construction, although being restricted to repelling invariant sets can be regarded as a step towards the understanding of statistical mechanics of space–time chaotic systems, in particular since explicit expressions relating the spin Hamiltonian and the parameters of the map lattice have been supplied.

Within analytical approaches the role of pruning for space–time periodic orbits has not been addressed so far. Variants of the proposed model class can be used to investigate these features since on has access to the corresponding statistical mechanics. For instance, it seems tempting to extend and test periodic orbit expansions for space–time chaotic systems.

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