

Some experimental approaches to Diophantine approximation problems

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Aim

- ★ To generalize some classical ideas from continued fractions to the problem of *simultaneous approximation* of sets of irrationals by rationals with common denominator ■
- ★ To understand *periodicity* questions related to approximation in number fields ■
- ★ To construct *badly approximable sets* ■
- ★ To study *statistics of blocks of digits* of continued fractions ■
- ★ To do all of the above by *computer experimentation*, when other approaches have failed

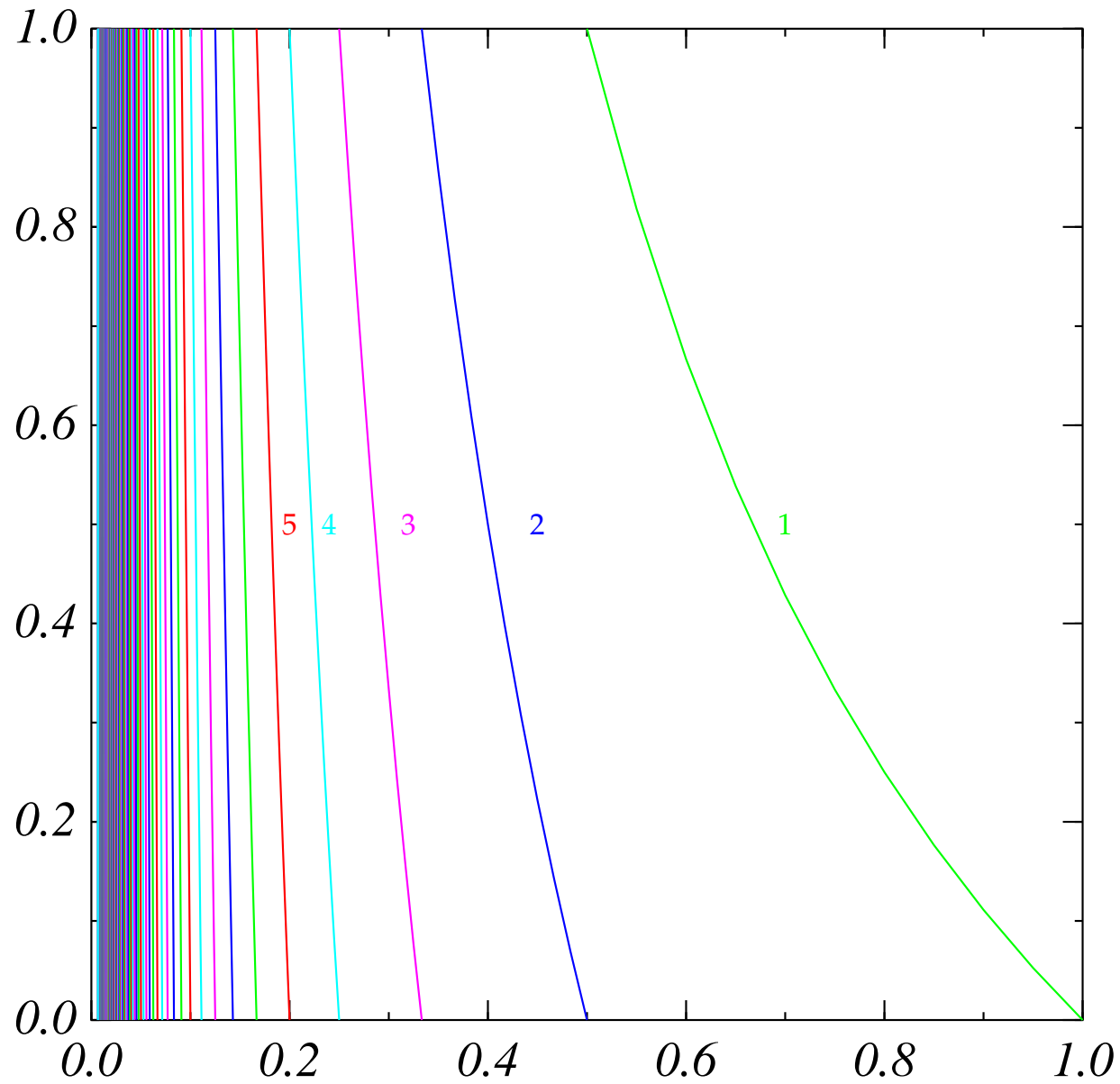
Continued fractions

- ★ *(Simple, regular) continued fractions* are symbolic dynamics of the Gauss map:

$$g(x) = 1/x - \lfloor 1/x \rfloor \quad \text{for } x \in (0, 1] \quad \blacksquare$$

- ★ The *partial quotient* ('digit') $x_k = \lfloor 1/g^{\langle k-1 \rangle}(x) \rfloor$ ($x_k \in \{1, 2, 3, \dots\}$) is output at the k th iteration \blacksquare
- ★ We write $x = [x_1, x_2, x_3, \dots] \equiv 1/(x_1 + 1/(x_2 + 1/(x_3 + \dots)))$ \blacksquare
- ★ The continued fraction is *finite* iff x is rational \blacksquare
- ★ The continued fraction is *eventually periodic* iff x is a quadratic irrational

Gauss map



Approximation properties of continued fractions

★ If (for $k = 1, 2, 3, \dots$)

$$\begin{aligned} \begin{bmatrix} p_{k-1} & q_{k-1} \\ p_k & q_k \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & x_k \end{bmatrix} \begin{bmatrix} p_{k-2} & q_{k-2} \\ p_{k-1} & q_{k-1} \end{bmatrix} \\ \begin{bmatrix} p_{-1} & q_{-1} \\ p_0 & q_0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

then $\frac{p_k}{q_k}$ is *precisely* the sequence of best approximants to x ■

★ That is, $|q_K x - p_K| < |q_k x - p_k| \quad \forall k < K$

★ In particular, $\frac{1}{(x_{k+1}+2)q_k} < |q_k x - p_k| < \frac{1}{x_{k+1}q_k}$ ■

★ Note for small denominator applications: if $\lambda = \exp(2\pi i\alpha)$, then $\forall q$ we have $4|\alpha q - \lfloor q\alpha \rfloor| < |\lambda^q - 1| < 2\pi|q\alpha - \lfloor q\alpha \rfloor|$

Diophantine approximation in one dimension

- ★ In one dimension, we measure the goodness of approximation of the rational number p/q to α by $c(\alpha, p, q) \equiv q|q\alpha - p|$ ■
- ★ For each irrational α there are infinitely many rationals p/q such that $|\alpha - p/q| < 1/q^2$; that is, $c(\alpha, p, q) < 1$ ■
- ★ The *approximation constant* of α is $c(\alpha) \equiv \liminf_{q \rightarrow \infty} c(\alpha, \lfloor q\alpha \rfloor, q)$ ■
- ★ Introducing the notation $\{ \alpha \}$ for the distance from α to the nearest integer, we have $c(\alpha) \equiv \liminf_{q \rightarrow \infty} q \{ q\alpha \}$ ■
- ★ α is said to be *badly approximable* if $c(\alpha) > 0$ ■
- ★ The *one-dimensional diophantine approximation constant* is $c_1 \equiv \limsup_{\alpha \in \mathbb{R}} c(\alpha)$, and this is known to have the value $1/\sqrt{5}$, attained for example at the golden ratio $\alpha = (\sqrt{5} - 1)/2$

Diophantine approximation in two dimensions

In the two-dimensional case, we wish to simultaneously approximate a pair of irrationals by a pair of rationals with common denominator:

★ The closeness of approximation is measured by the *maximum* error of the two components ■

★ We thus extend the meaning of the symbol $\{ \cdot \}$ by
 $\{ \alpha \} \equiv \min_{p \in \mathbb{Z}^2} \max(|q\alpha_1 - p_1|, |q\alpha_2 - p_2|)$ ■

★ For $q \in \mathbb{Z}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, let

$$c(\alpha, q) = q \{q\alpha\}^2, \quad c(\alpha) = \liminf_{q \rightarrow \infty} c(\alpha, q) \quad \blacksquare$$

★ The *two-dimensional sup-norm simultaneous diophantine approximation constant* is then $c_2 = \sup_{\alpha} c(\alpha)$ ■

★ The value of c_2 is unknown, but bounds are known: $2/7 < c_2 < 64/169$

Algebraic number fields

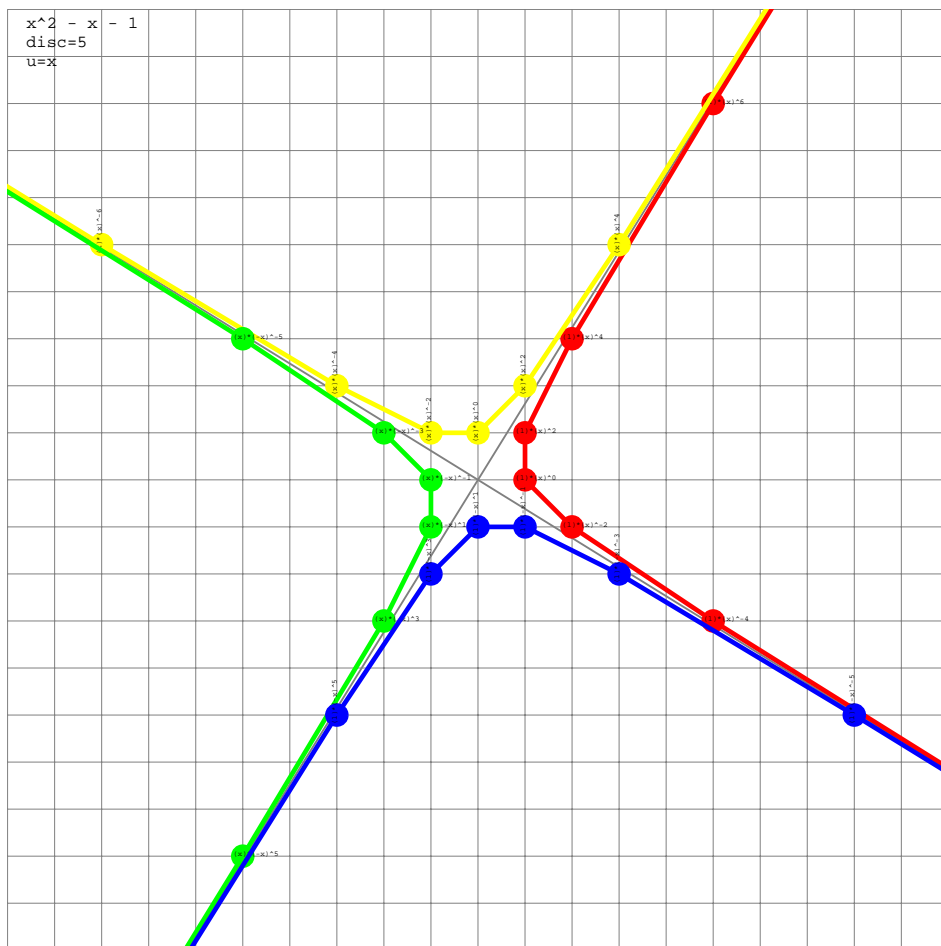
- ▶ K is a *number field* $\Leftrightarrow \mathbb{Q} \subset K$ and K is a finite-dimensional vector space over \mathbb{Q} ■
- ▶ if $K = \mathbb{Q}(\theta)$, θ is called a *primitive element* ■
- ▶ The *conjugates* of θ are the roots θ_i of the minimal polynomial of θ . These define n embeddings of K in \mathbb{C} ■
- ▶ Each K has a *discriminant* which is a squared multiple of the discriminant of the minimal polynomial of θ ■
- ▶ The *norm* Nx of an element x is the product of x with its conjugates ■
- ▶ A *unit* is an element of norm ± 1 . The set of units $U(K)$ forms a multiplicative group ■
- ▶ *Dirichlet's theorem*: there exists a set of units $\{u_1, \dots, u_r\}$ such that every unit u can be expressed as $u = \zeta u_1^{n_1} \cdots u_r^{n_r}$, where ζ is a root of unity ■
- ▶ Such a set is called a *set of fundamental units* ■
- ▶ In other words, $U(K)$ is isomorphic to a product of a cyclic group and an additive Abelian group

Algebraic number fields cotsd.

- ▷ $x \in K$ is an *algebraic integer* if it is the root of a monic polynomial $f \in \mathbb{Z}[x]$ ■
- ▷ The set of algebraic integers in K form a ring \mathbb{Z}_K ■
- ▷ A \mathbb{Z} -basis of \mathbb{Z}_K considered as a \mathbb{Z} -module is called an *integral basis* ■

polynomial	field	d	integral basis	fund. units
$x^2 - x - 1$	$\mathbb{Q}(\sqrt{5})$	5	$[1, x]$	$\{x\}$
$x^2 - 5$	$\mathbb{Q}(\sqrt{5})$	5	$[1, (x+1)/2]$	$\{(x+1)/2\}$
$x^2 - 2$	$\mathbb{Q}(\sqrt{2})$	8	$[1, x]$	$\{x-1\}$
$x^2 - 3$	$\mathbb{Q}(\sqrt{3})$	12	$[1, x]$	$\{x-2\}$
$x^3 + 2x^2 - x - 1$	$\mathbb{Q}(2 \cos(\frac{2\pi}{7}))$	49	$[1, x, x^2]$	$\{x^2 - 1, x + 1\}$
$x^3 - x - 1$	$\mathbb{Q}(?)$	-23	$[1, x, x^2 - 1]$	$\{x\}$

Klein polygons - $\mathbb{Q}(\sqrt{5})$



$K = (\text{Field of } x^2 - x - 1) = \mathbb{Q}(\sqrt{5})$

Fundamental unit: $u = x$

Norm of f.u.: $Nu = -1$

Integral basis: $B = \{1, x\}$

$(\sqrt{5} - 1)/2 = [\bar{1}]$

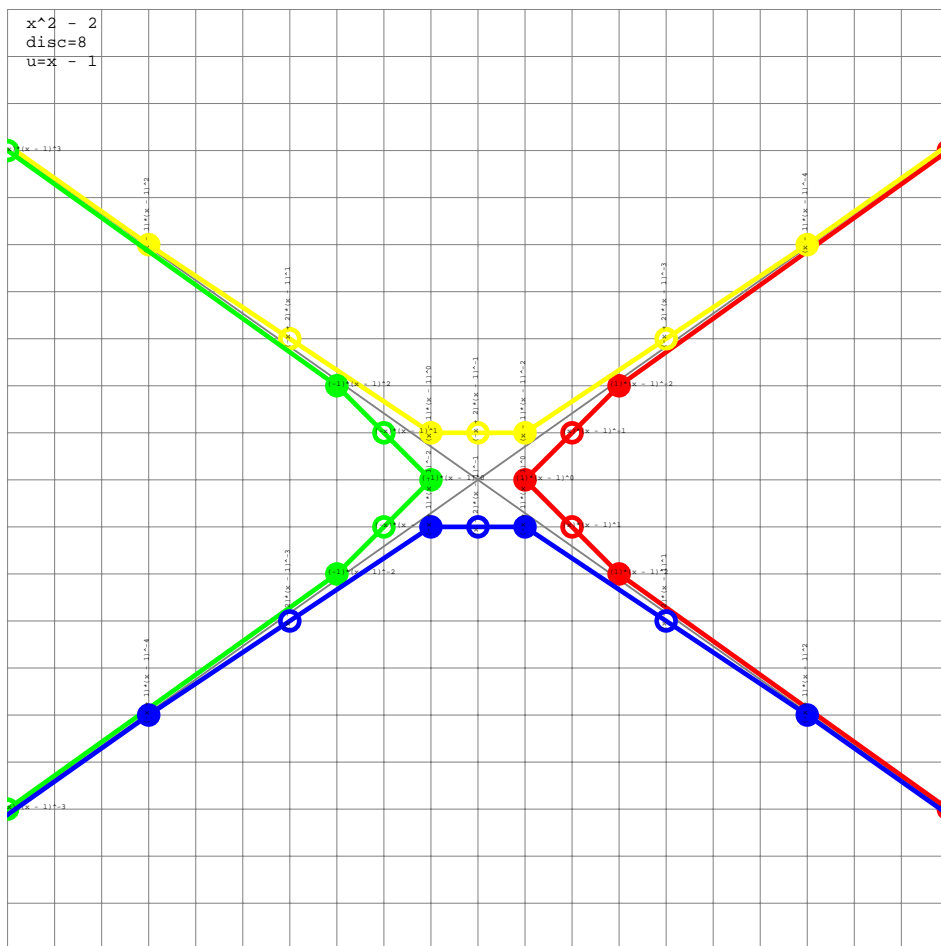
$u^{2\mathbb{Z}}$

$u^{2\mathbb{Z}-1}$

$-u^{2\mathbb{Z}}$

$-u^{2\mathbb{Z}-1}$

Klein polygons - $\mathbb{Q}(\sqrt{2})$



$K = (\text{Field of } x^2 - 2) = \mathbb{Q}(\sqrt{2})$

Fundamental unit: $u = x - 1$

Norm of f.u.: $Nu = -1$

Integral basis: $B = \{1, x\}$

$\sqrt{2} = [1, \bar{2}]$

$u^{2\mathbb{Z}}$

$u^{2\mathbb{Z}-1}$

$-u^{2\mathbb{Z}}$

$-u^{2\mathbb{Z}-1}$

Klein polyhedra

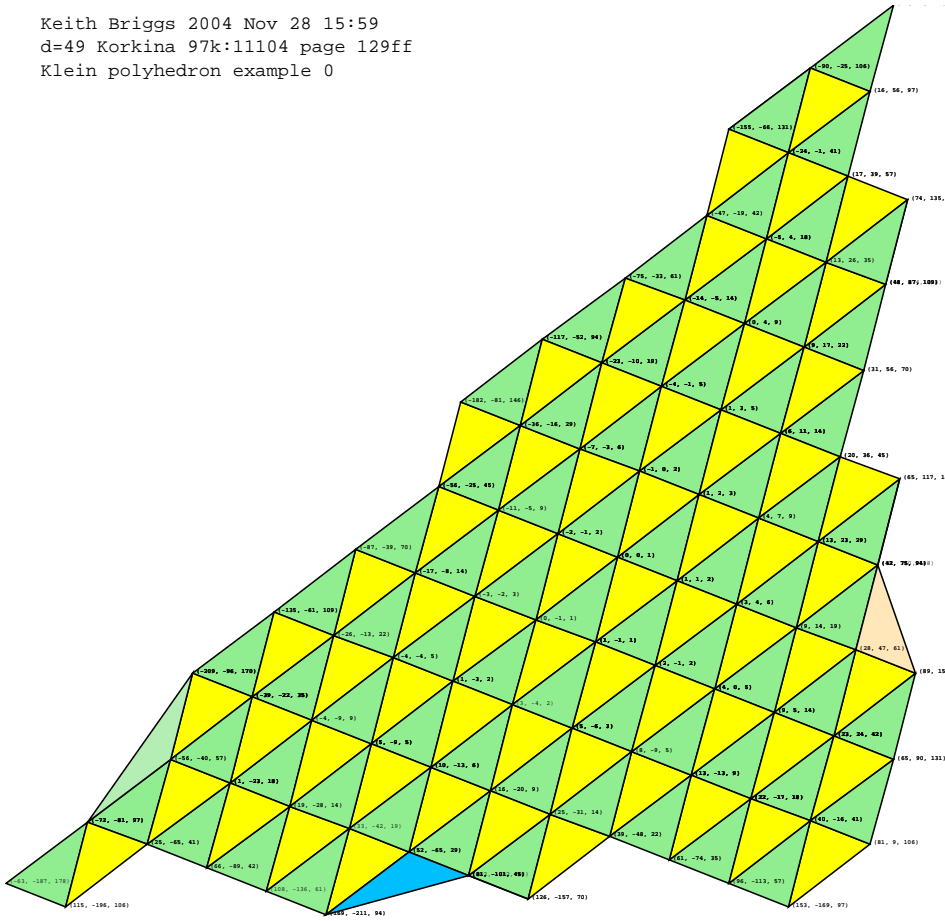
- ★ We work in \mathbb{R}^3 , and in the lattice \mathbb{Z}^3 embedded in it
- ★ Consider three planes with (not necessarily unit) normals α_i ($i = 1, 2, 3$) through the origin. Consider the octant defined by $\alpha_i(x) > 0$ ■
- ★ Now form the convex hull of the points of \mathbb{Z}^3 (excluding the origin) contained in this octant. This is the *Klein polyhedron* of the cubic form $\alpha(x) = \alpha_1(x)\alpha_2(x)\alpha_3(x)$ ■
- ★ To visualize a Klein polyhedron, we take a finite piece near the origin and flatten the piece onto a plane. More precisely, for each vertex point x in the convex hull, we let $y = |\alpha(x)|^{-1/3}x$ and plot (z_1, z_2) , where

$$z = (\log |\alpha_1(y)|, \log |\alpha_2(y)|, \log |\alpha_3(y)|) \blacksquare$$

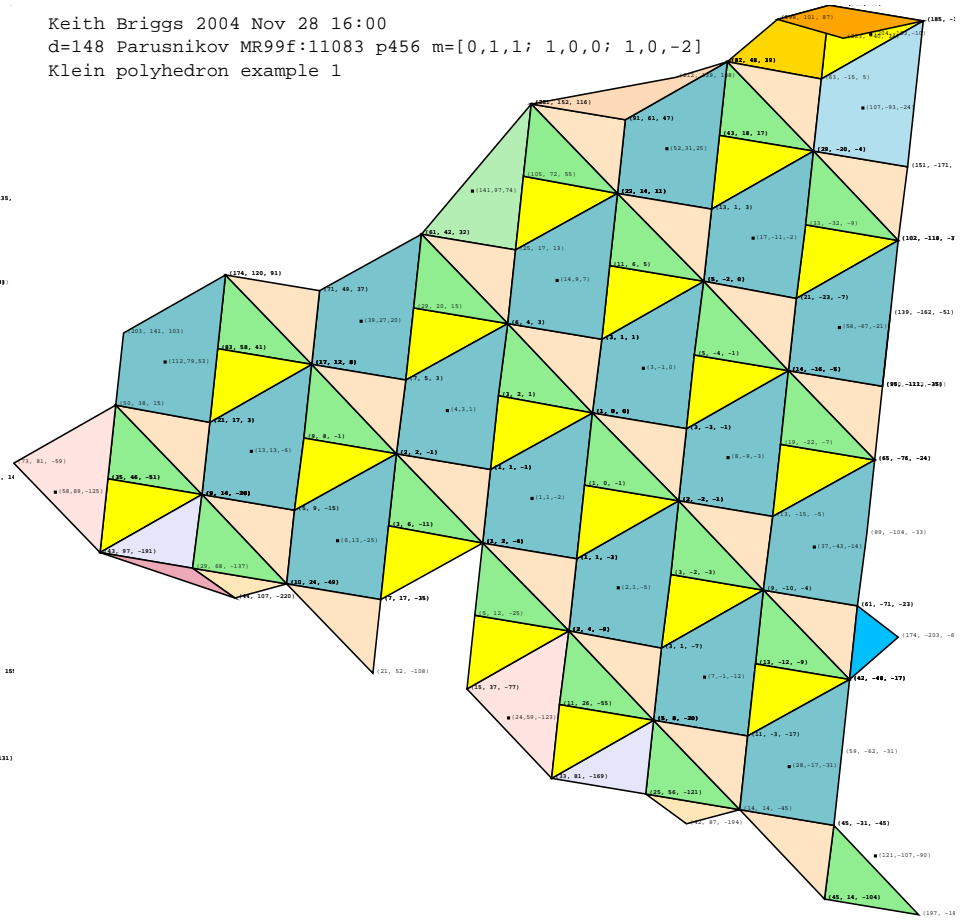
- ★ Thus $z_1 + z_2 + z_3 = 0$, so by plotting any two components we have full information ■
- ★ The main point of interest is that the patterns are periodic iff the planes are related to a totally real cubic number field

Klein polyhedra examples - $d = 49$ and $d = 148$

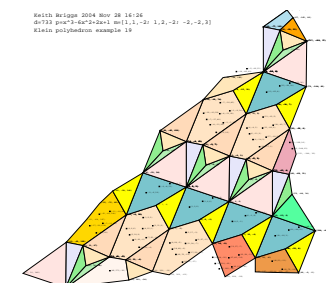
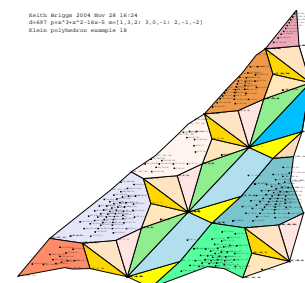
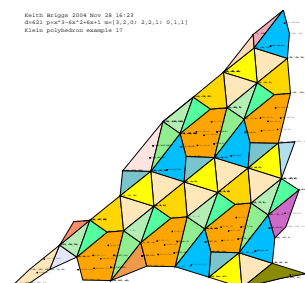
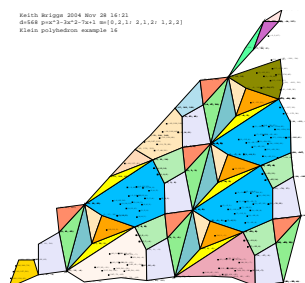
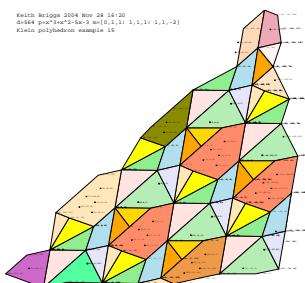
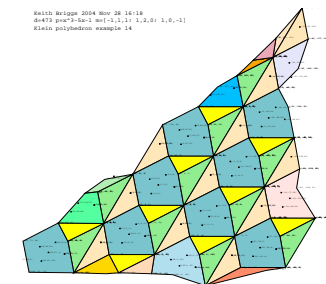
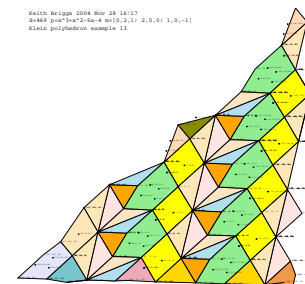
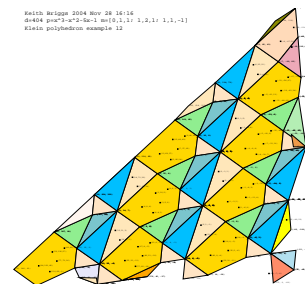
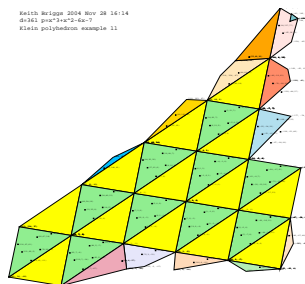
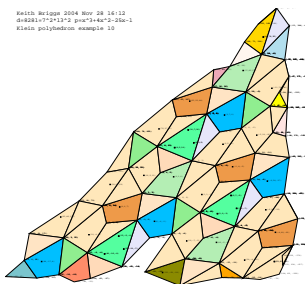
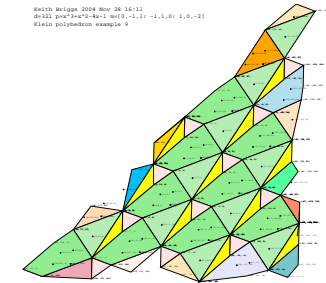
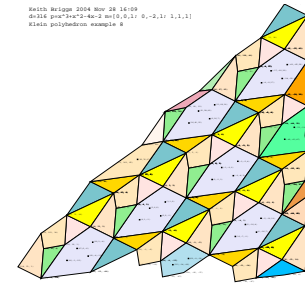
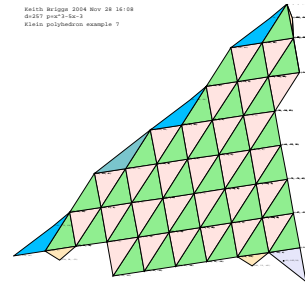
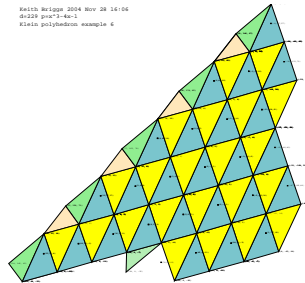
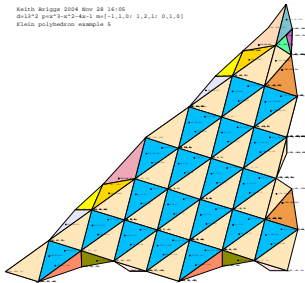
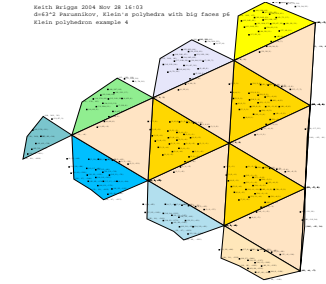
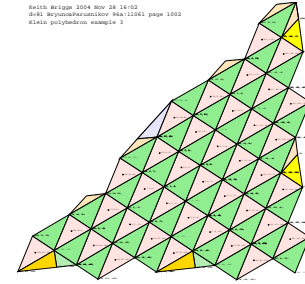
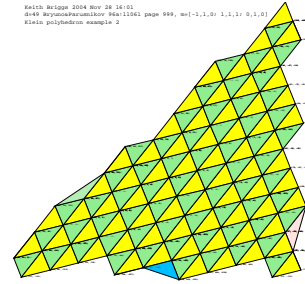
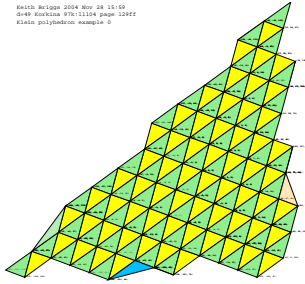
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d=49 Korkina 97k:11104 page 129ff
Klein polyhedron example 0



Keith Briggs 2004 Nov 28 16:00
d=148 Parusnikov MR99f:11083 p456 m=[0,1,1; 1,0,0; 1,0,-2]
Klein polyhedron example 1



More Klein polyhedra examples



The Furtwängler algorithm

★ The aim of the Furtwängler algorithm [Fur26, Fur28] is to find all best sup-norm simultaneous rational approximations to a given irrational pair $(\alpha_1, \alpha_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ ■

★ $(Q, P_1, P_2) \in \mathbb{Z}^3$ is called a *best approximation triple* if $\forall q < Q$

$$\max(|q\alpha_1 - p_1|, |q\alpha_2 - p_2|) > \max(|Q\alpha_1 - P_1|, |Q\alpha_2 - P_2|) \blacksquare$$

- ▷ *The algorithm works by keeping an approximation matrix A whose rows we label P, Q, R . A typical row contains integers (p_1, p_2, q) corresponding to an approximant $(p_1/q, p_2/q)$, which need not be a best approximant. A step of the main loop of the algorithm consists of finding a new row S which will replace one of P, Q or R , and a reordering of the rows. The new row S is always an integer linear combination of P, Q and R ■*
- ▷ *Choosing S involves considering several possibilities, and the final choice is such that no best approximant will be missed, though the number of iterations between best approximations may be arbitrarily large*
- ▷ *there are five different unimodular update matrices B (with $a, b \in \mathbb{Z}$):*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

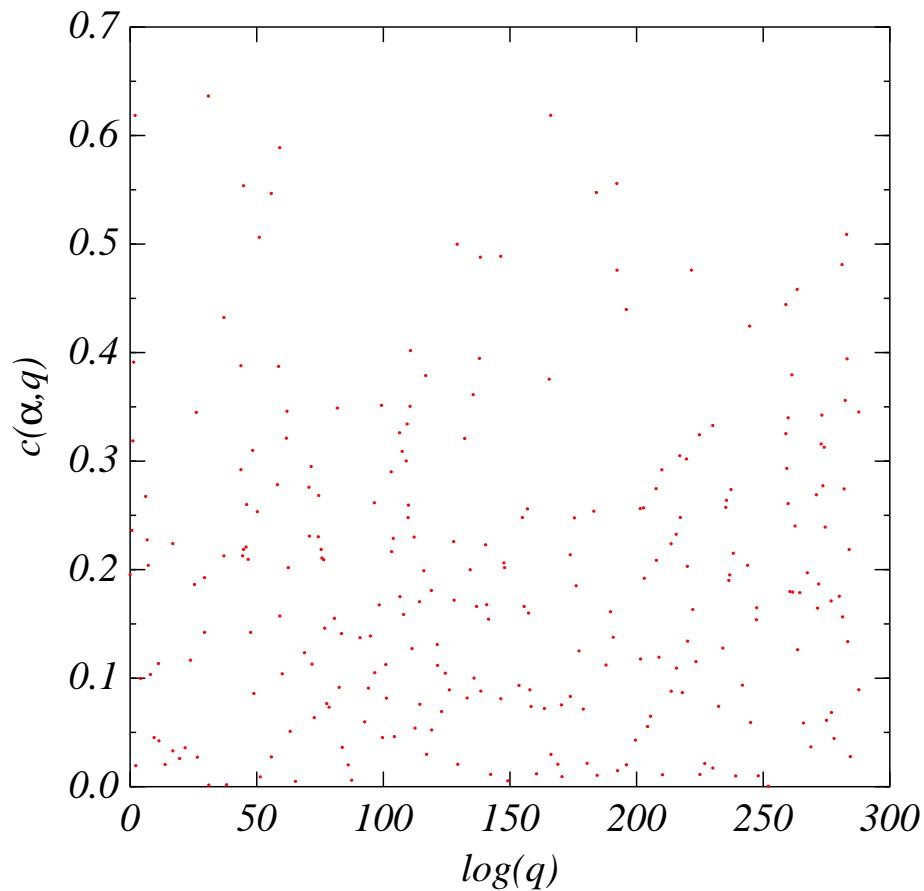
$$\begin{bmatrix} 1 & 0 & 0 \\ a & b & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

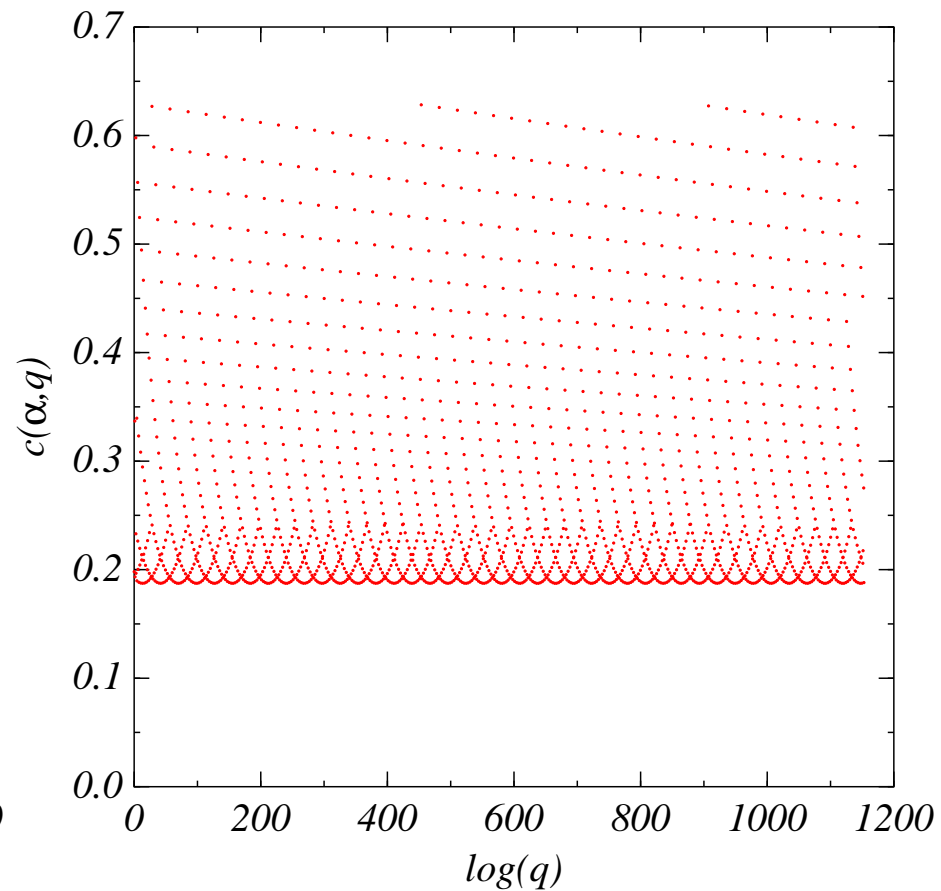
$$\begin{bmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The Furtwängler algorithm - behaviour

$d=+49$

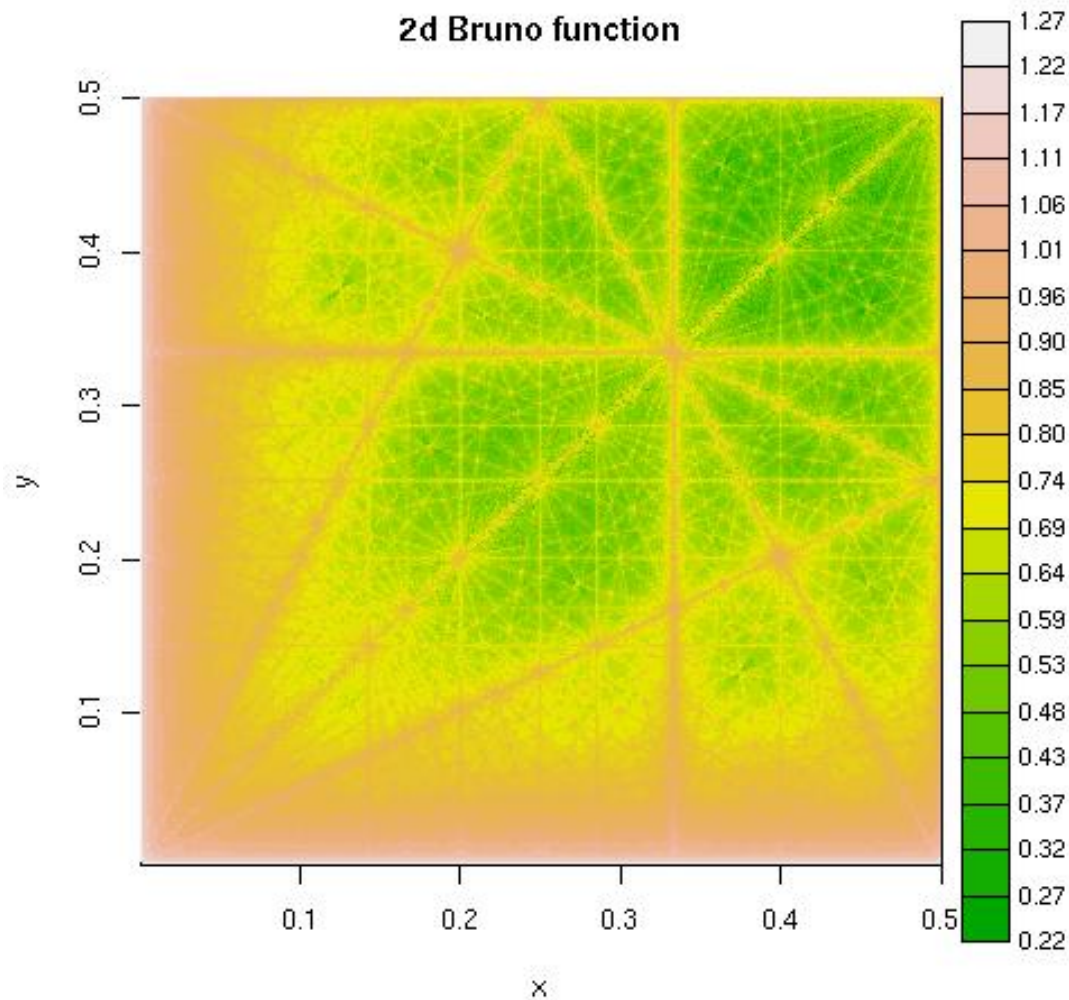


$d=+49$



$c(\alpha, q)$ at best approximation denominators q . Left: α = a 'random' pair of irrationals. Right: $\alpha = (4t^2 - 1, 2t - 1)$, $t = \cos(2\pi/7)$.

The 2-dimensional Bruno function



$$B(\alpha) \equiv \sum_{i=0}^{\infty} \log(q_{i+1})/q_i$$

where $\{q_0, q_1, \dots\}$ are the
BSADs of $\alpha = (x, y)$

The worst approximable pair

★ In [Bri03], I used some theorems of Cusick to explicitly construct some provably badly approximable pairs ■

★ The worst pair found was

$$\alpha_1 \approx 0.4848739572889332951989678247806190621159456336657613$$

$$\alpha_2 \approx 0.5404925035004667478257428539575752367424111926723566$$

which has $c_2(\alpha) > 0.2857082$ ■

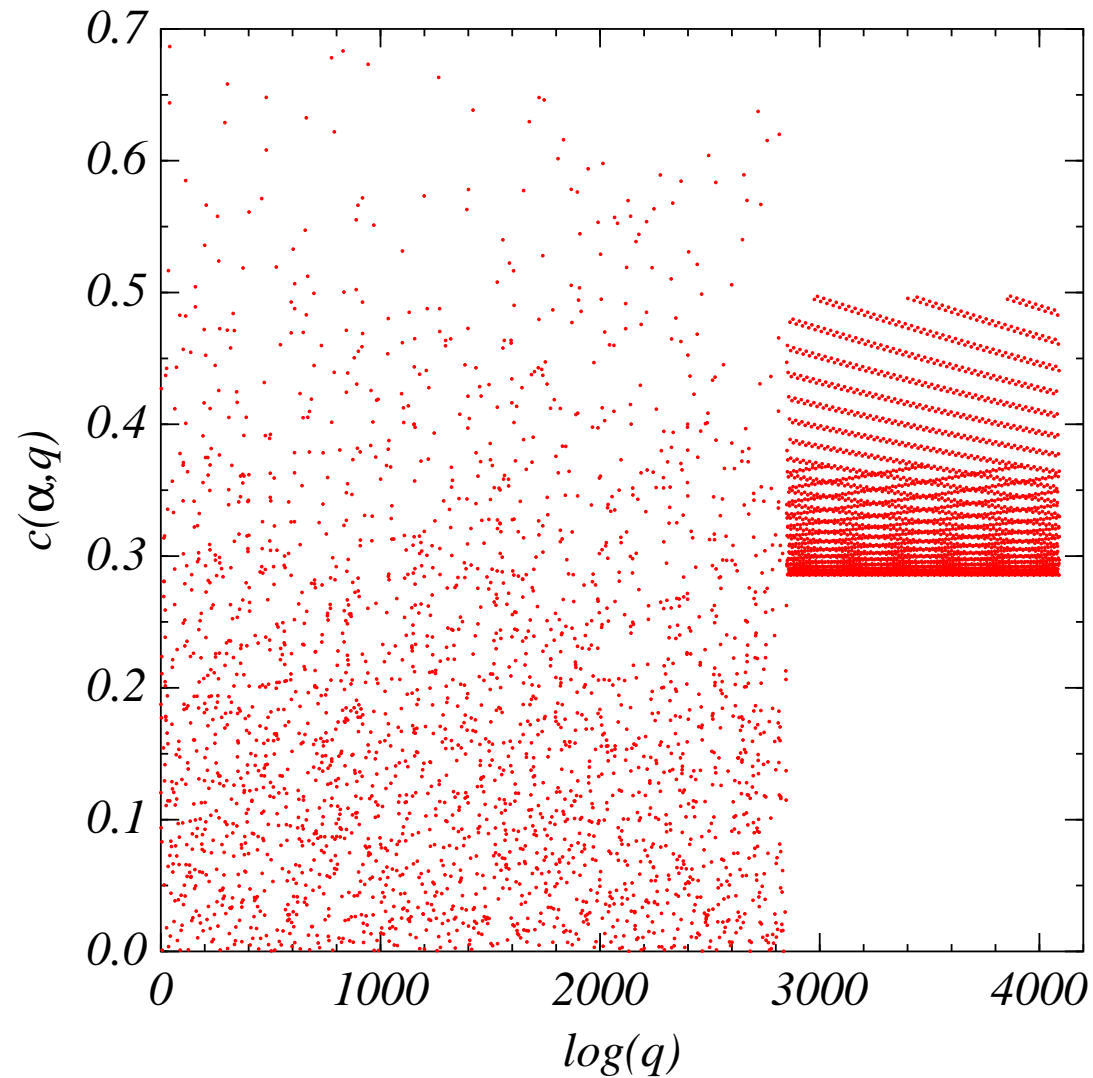
★ To fully specify this pair would take several million bits ■

★ The method depends on finding sequences in the continued fraction of $2 \cos(2\pi/7)$ of the form $[\dots, n_1, 1, 1, n_2, \dots]$ with n_1, n_2 large ■

★ It is not known whether n_1, n_2 become arbitrarily large. If so, $c_2 = 2/7$ can be obtained in this field

The worst approximable pair in Furtwängler's algorithm

the worst pair



The \mathbb{Z}^4 map for partial quotients of $2 \cos(2\pi/7)$

★ By a method of Lagrange, the partial quotients of $2 \cos(2\pi/7)$ are given by the output of this map on \mathbb{Z}^4 : ■

★ **Algorithm:**

▷ **Initialize:**

$$z \leftarrow [-1, -2, 1, 1] = [z_0, z_1, z_2, z_3]$$

▷ **repeat:**

$$w \leftarrow 1$$

$$\text{for } j \text{ in } 2, 1, 0: z_j += z_{j+1}$$

$$\text{for } j \text{ in } 2, 1: z_j += z_{j+1}$$

$$\text{for } j \text{ in } 2: z_j += z_{j+1}$$

$$\text{if } z_0 + z_1 + z_2 + z_3 < 0: w += 1$$

$$\text{else: output } w; z \leftarrow [-z_3, -z_2, -z_1, -z_0]$$

Ergodic properties of continued fractions

For almost all irrational x , ordinary 1d continued fractions have these properties as a consequence of the invariant measure of the Gauss map being $\log_2(1+x)$:

★ the digit i occurs with relative frequency $\mu(i) \equiv \log_2 \left[\frac{(i+1)^2}{i(i+2)} \right]$ ■

★ $\lim_{k \rightarrow \infty} (x_1 x_2 x_3 \dots x_k)^{1/k} = 2.68545 \dots$ ■

★ The *denominator growth rate* is

$$g_1 \equiv \lim_{k \rightarrow \infty} q_k^{1/k} = \exp(\pi^2 / (12 \log 2)) = 3.27582 \dots \quad \blacksquare$$

Lagarias has proven that g_2 , the growth rate of 2d BSADs is bounded by $g_2 > \sqrt{\frac{1+\sqrt{5}}{2}} = 1.27202 \dots$ ■

★ Using Furtwängler's algorithm, I estimated that the average g_2 is really about 3.07 ■

★ I found the smallest g_2 in the field of discriminant -23 , where $g_2 \approx 1.563$

Statistics of blocks for 1d continued fractions

- ★ I will look at occurrences of *finite blocks* of digits $i = (i_1, i_2, \dots, i_m), i_j \geq 1$ ■
- ★ [IK02] gives a formula for relative frequency of the m -block i which holds $\forall \epsilon > 0$ as $n \rightarrow \infty$ for almost all irrationals:

$$\text{card}\{\kappa : (x_\kappa, \dots, x_{\kappa+m-1}) = i, 1 \leq \kappa \leq n\} / n = \log_2 \left[\frac{1+v(i)}{1+u(i)} \right] + o\left(n^{-1/2} \log^{(3+\epsilon)/2}(n)\right)$$

where (with $[i] = p_m/q_m$ for the m -block i)

$$u(i) = \begin{cases} (p_m + p_{m-1}) / (q_m + q_{m-1}) & \text{if } m \text{ is odd} \\ p_m / q_m & \text{if } m \text{ is even} \end{cases}$$

$$v(i) = \begin{cases} p_m / q_m & \text{if } m \text{ is odd} \\ (p_m + p_{m-1}) / (q_m + q_{m-1}) & \text{if } m \text{ is even} \end{cases}$$

Numerical values for the frequencies

For 2-blocks:

	1	2	3	4	5	6
1	0.15200	0.07038	0.04064	0.02647	0.01861	0.01380
2	0.07038	0.02914	0.01594	0.01005	0.00691	0.00505
3	0.04064	0.01594	0.00851	0.00529	0.00361	0.00262
4	0.02647	0.01005	0.00529	0.00326	0.00221	0.00160
5	0.01861	0.00691	0.00361	0.00221	0.00150	0.00108
6	0.01380	0.00505	0.00262	0.00160	0.00108	0.00078

Explicit examples of abnormal numbers

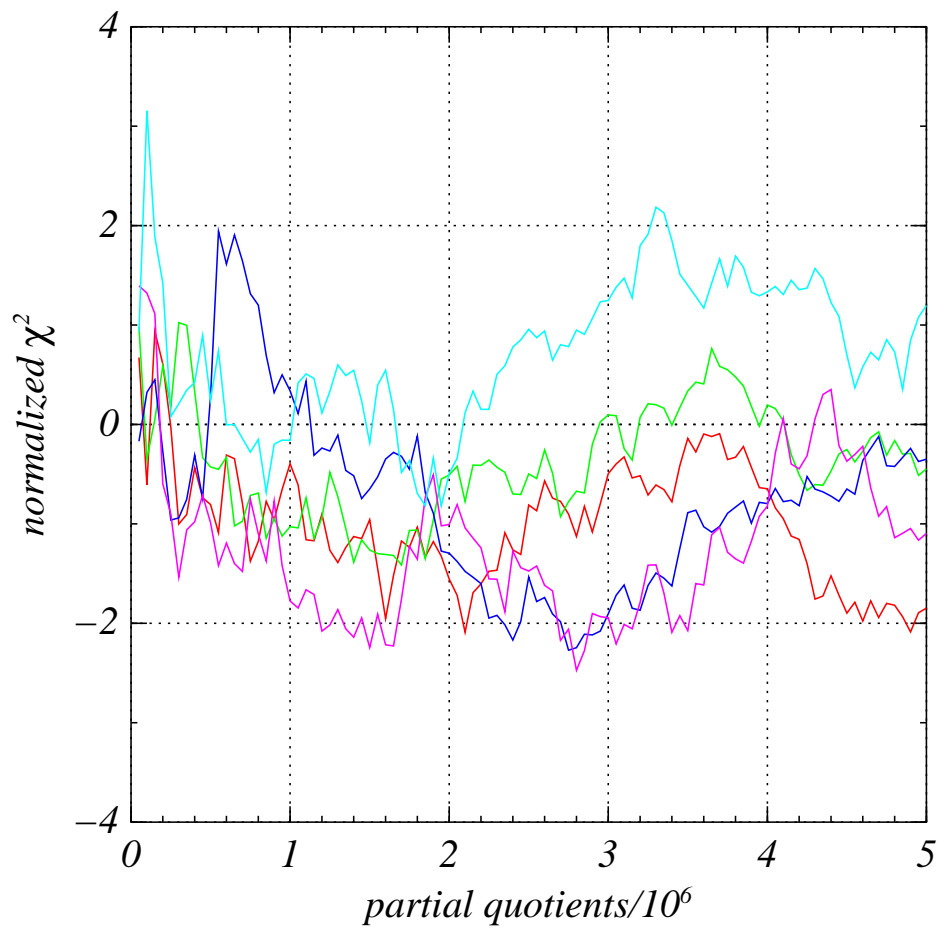
- ★ all quadratic irrationals, e.g. $2^{1/2} = 1 + [2, 2, 2, 2, \dots]$
- ★ $I_1(2)/I_0(2) = [1, 2, 3, 4, \dots]$ (ratio of modified Bessel functions)
- ★ $I_{1+a/d}(2/d)/I_{a/d}(2/d) = [a+d, a+2d, a+3d, \dots]$
- ★ $\tanh(1) = [1, 3, 5, 7, \dots]$
- ★ $\exp(1/n) = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, \dots]$; $n = 1, 2, 3, \dots$
- ★ $\exp(2) = 7 + [2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, 1, 1, \dots]$
- ★ $\exp(2/(2n+1))$; $n = 1, 2, 3, \dots$
- ★ $\sum_{k=1}^{\infty} 2^{-\lfloor k\phi \rfloor} = [2^0, 2^1, 2^1, 2^3, 2^5, 2^8, 2^{13}, \dots]$; $\phi = (\sqrt{5}-1)/2$

Method

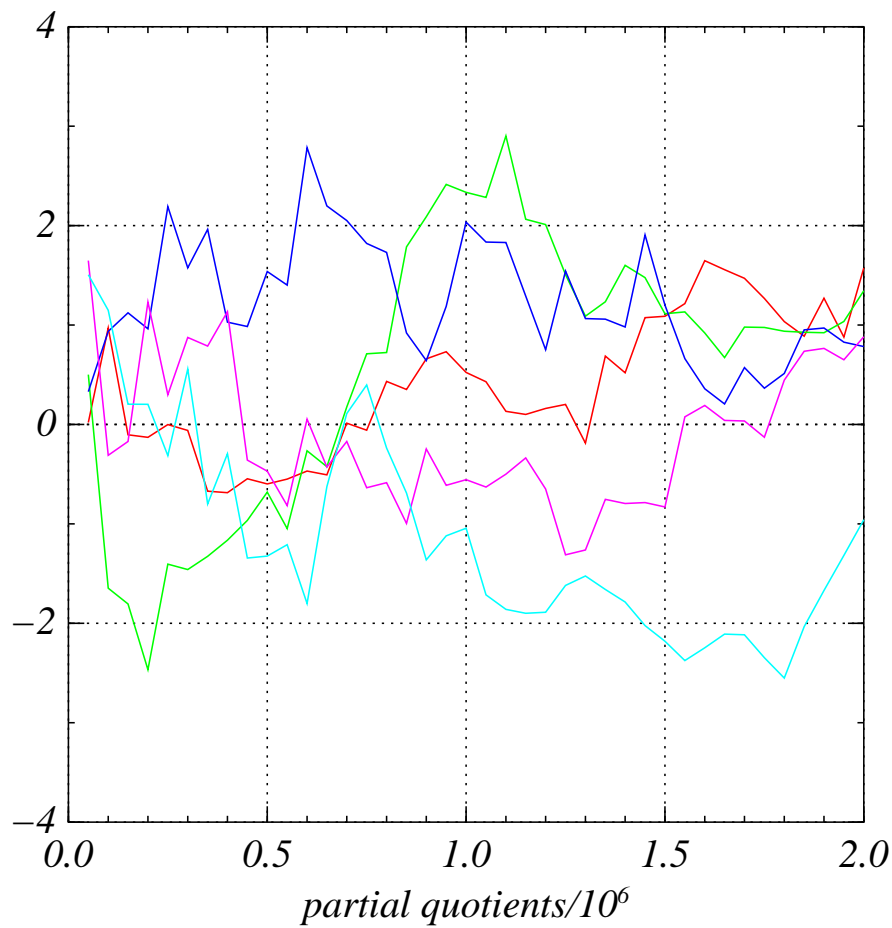
- ★ I calculated a few million digits for several cubic irrationals and a few other irrationals ■
- ★ I counted exactly the observed frequency of all blocks of lengths 1,2,3,4, and 5 ■
- ★ I calculated a Pearson χ^2 test statistic which measures the deviation of the observed frequencies from the expected frequencies ■
- ★ Because the number of degrees of freedom ν is so large (typically several thousand), a normal approximation is sufficiently accurate. The transformation is $Z \equiv \sqrt{2\chi^2} - \sqrt{2\nu - 1}$. Under the assumption of normality (of the cf of $x!$), Z is distributed $N(0, 1)$

χ^2 results: $2^{1/3}$ and $3^{1/3}$

cbrt2



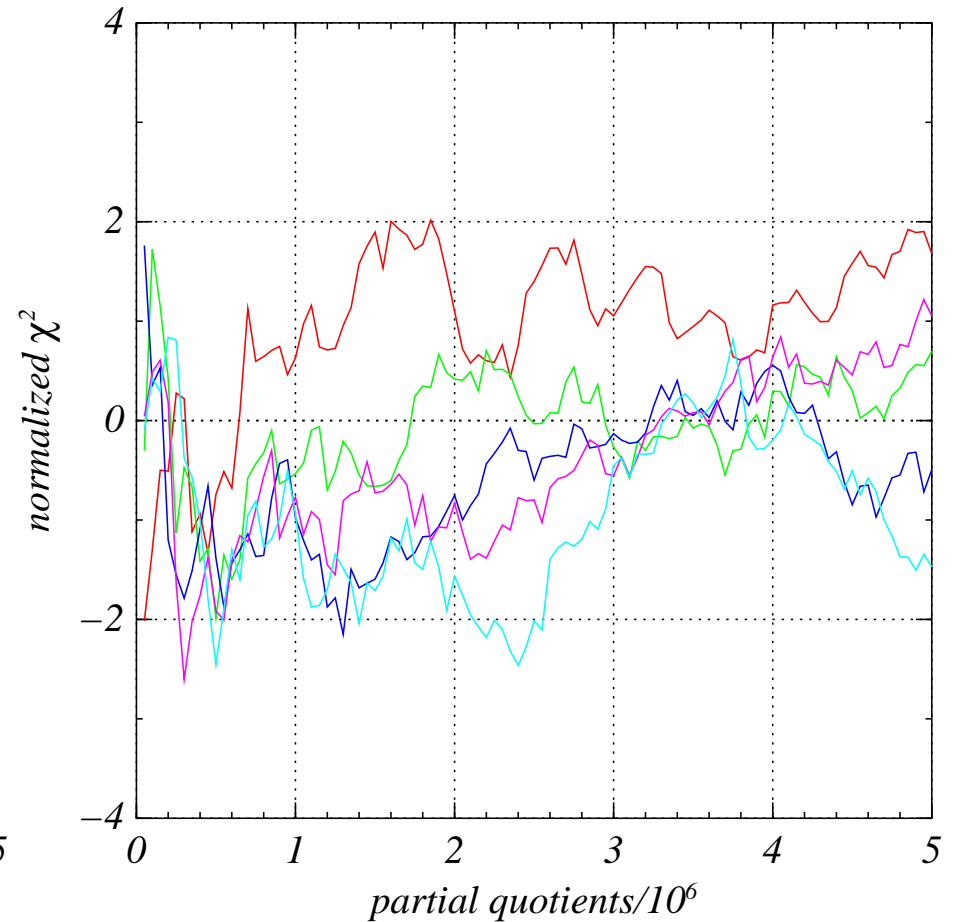
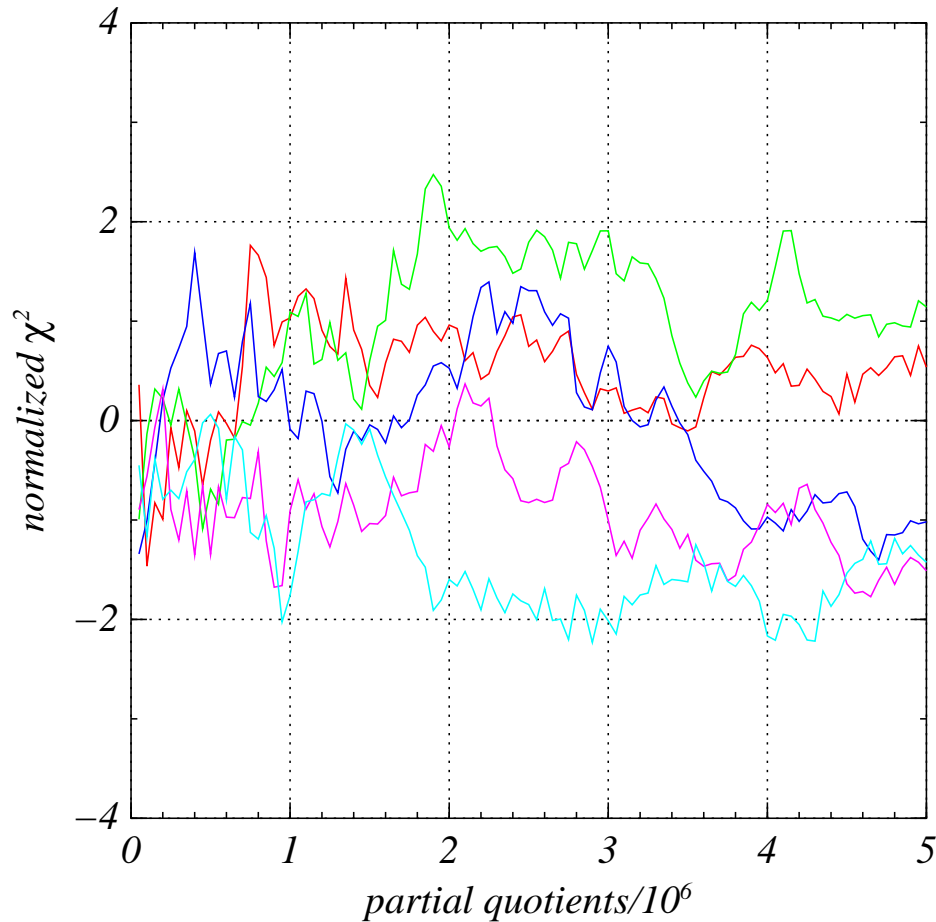
cbrt3



χ^2 results: $2 \cos(2\pi/7)$ and largest root of $x^3 - 8x - 10$

$2\cos 2\pi n 7$

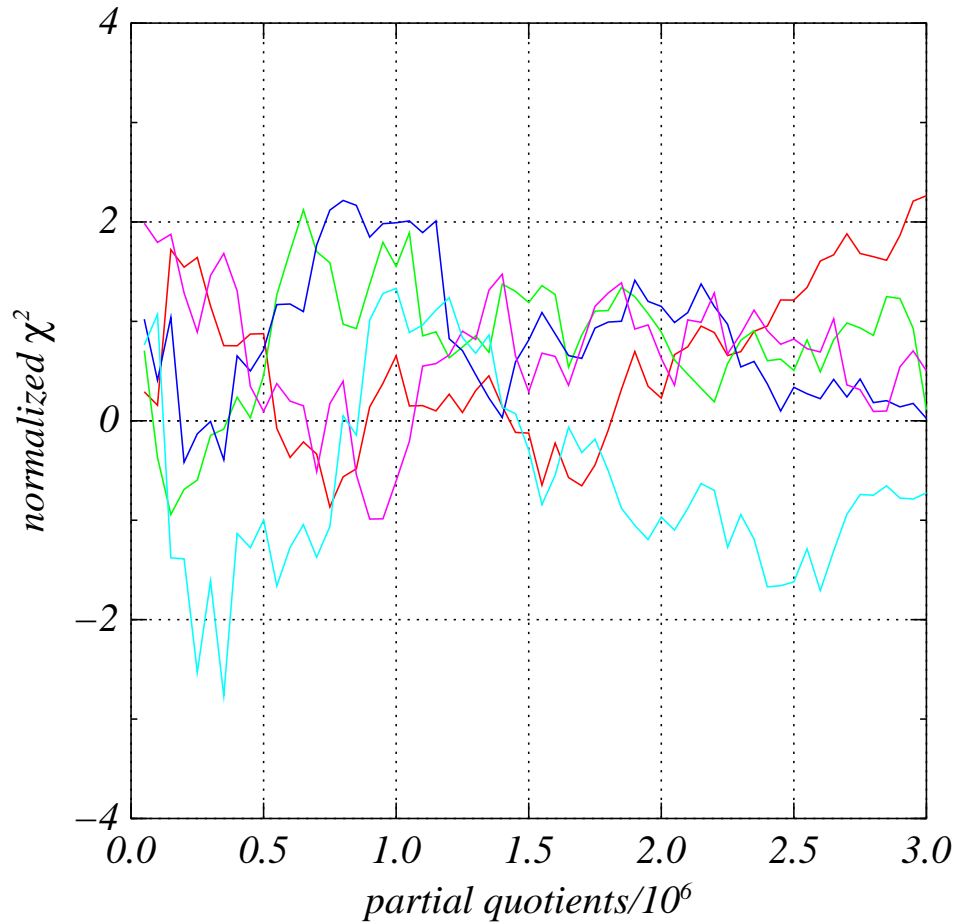
$m163$



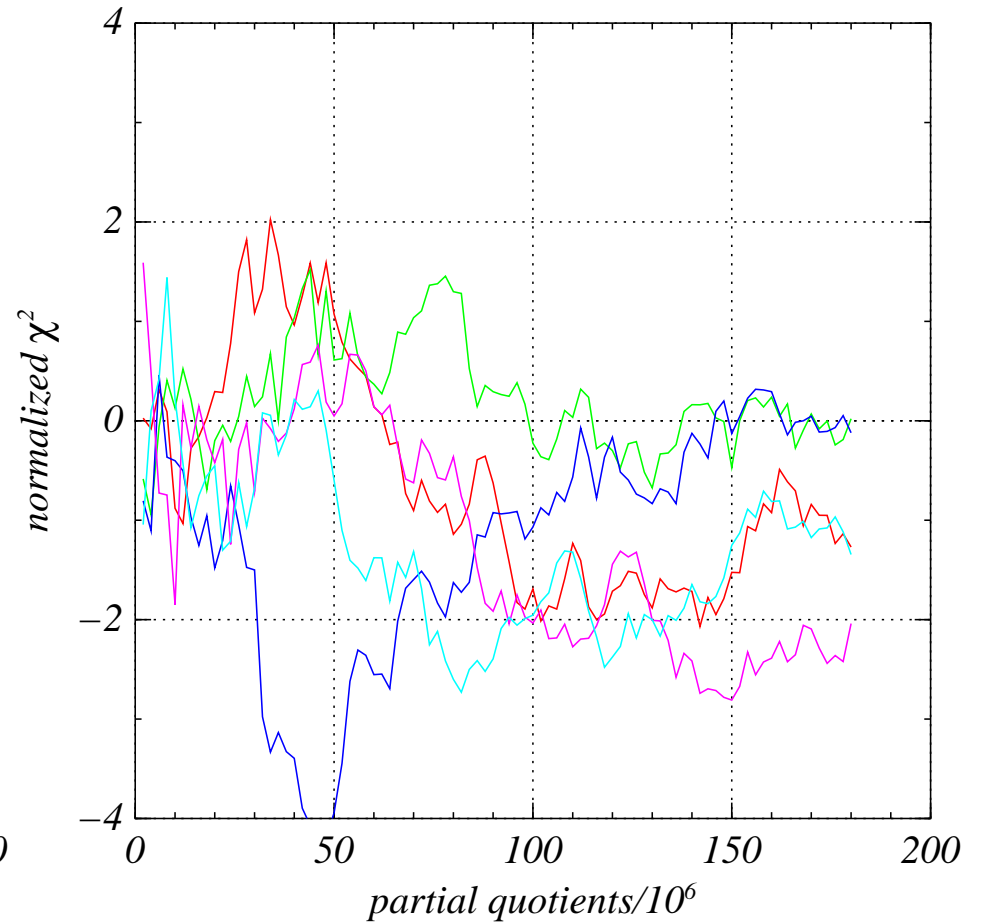
(the last example is famous for having several abnormally large digits)

χ^2 results: $(\sqrt{5}-1)/2+\sqrt{2}-1$ and π

1600



π



Autocorrelation of digits

- ★ We would expect that the autocorrelation function (acf) of any analytic function of the digits that has a finite mean (for example, the log or the reciprocal) would decay like q^k at lag k , where $q \approx -0.303663$ is *Wirsing's constant* ■
- ★ This is investigated in the following graphs. I plot \log_{10} of the absolute value of the acf as a function of lag. The green line has the Wirsing slope ■
- ★ In Rockett & Szűsz [RS92], we have the result

$$\Pr [x_n = r \ \& \ x_{n+k} = s] = \Pr [x_n = r] \Pr [x_{n+k} = s] (1 + O(q^k))$$

This, however, is too weak to allow explicit statistical tests

acf estimation difficulties

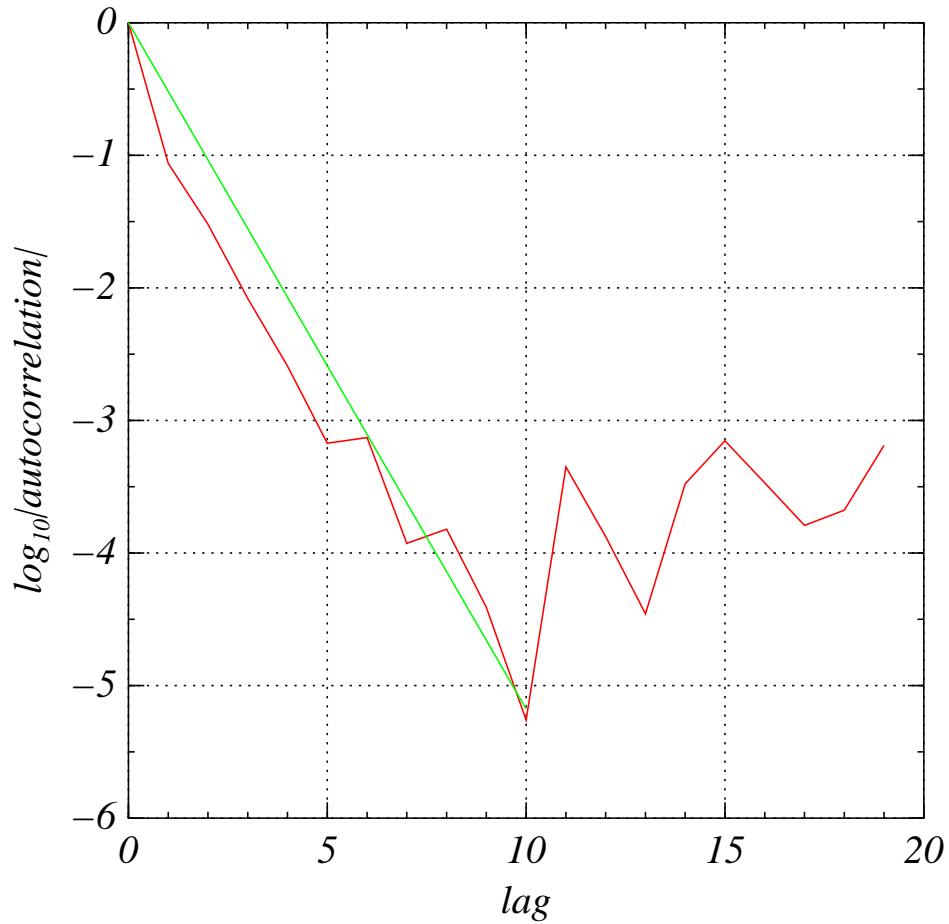
- ★ For the AR(1) process $x(t+1) = \alpha x(t) + \epsilon$, $|\alpha| < 1$, the exact acf at lag k is $\rho(k) = \alpha^k$ ■
- ★ But the usual acf estimator r for a sample of size n has variance

$$\text{var} [r_n(k)] = \frac{1}{n} \left[\frac{(1+\alpha^2)(1+\alpha^{2k})}{1-\alpha^2} - 2k\alpha^{2k} \right] \quad \blacksquare$$

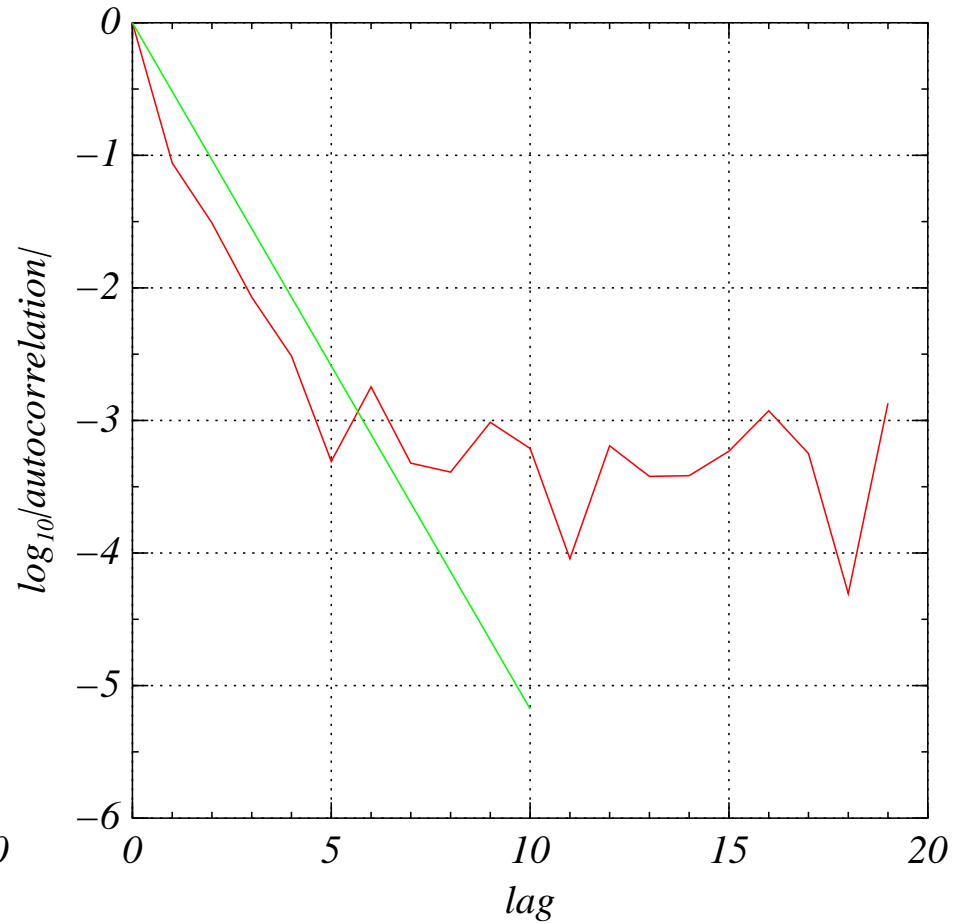
- ★ More generally, for a process whose acf decays for large k in the same power-law fashion, we have approximate variance $\text{var} [r_n(k)] = \frac{1}{n} \left[\frac{1+\alpha^2}{1-\alpha^2} \right]$ for large k ■
- ★ I expect my process to conform to this behaviour, and if it does, putting in the numbers gives an estimate of $k = 6$ for the largest k for which the acf estimates are meaningful 🏆

autocorrelation of logs of digits: $2^{1/3}$ and $3^{1/3}$

cbrt2



cbrt3



Appendix: history 1

- ▶ *In the early 1900s, Minkowski's work on geometry of numbers [Min11b, Min11a] provided a theoretical basis for all subsequent work. Perron, Klein etc. established the metric theory of one-dimensional continued fractions.*
- ▶ *In the 1920s, Furtwängler published the first algorithm which aimed to find all the best sup-norm approximants up to a given denominator. This work was largely forgotten [Fur26, Fur28, Bri01].*
- ▶ *1950s: Davenport, Cassels.*
- ▶ *In 1970, G. Szekeres [Sze70] published his multi-dimensional continued fraction algorithm. It is now known that this does not find all best sup-norm approximants.*
- ▶ *In the 1970s there was significant work by Cusick, Adams and Krass (see bibliography) especially on the relation of two-dimensional approximation to cubic number fields.*
- ▶ *In 1981, Brentjes completed his thesis, which made major contributions to the field, in particular to two-dimensional Euclidean norm approximation. The published version of the thesis is now a basic reference in the field ([Bre81])*

Appendix continued

- ▶ Around 1985, G. Szekeres published three papers on computer experiments intended to search for the 2 and 3 dimensional approximation constants. This work was not fully rigorous but gave intriguing results which have never been followed up.
- ▶ [Sch95] studied ergodic properties of various algorithms.
- ▶ In the 1990s several Russian and French mathematicians developed the concept of Klein polyhedra ([Lac93, BP94, Kor94, Kor95, BP97, Lac98a, Lac98b, Arn98, KS99])
- ▶ In 1995, Lagarias and Pollington published a clear analysis of the Szekeres multi-dimensional continued fraction algorithm ([LP95])
- ▶ In 1997, Clarkson ([Cla97]) completed his thesis in which was presented for the first time an algorithm provably finding all best approximants in two dimensions with respect to arbitrary radius and height functions.
- ▶ In the late 1990s, Khanin and Hardcastle proved some results about an n -dimensional Gauss map [HK00a, HK00b].
- ▶ A very good summary of the state-of-the-art is [Mos99]. See also [Sch96].

References

- [Arn98] V. I. Arnold. Higher dimensional continued fractions. *Regular and chaotic dynamics*, 3(3):10–17, 1998. MR 2000h:11012.
- [BP94] A. D. Bryuno and V. I. Parusnikov. Klein polyhedra for two cubic Davenport forms. *Mathematical notes*, 56(3-4):9–27, 1994. Keldysh Institute of the RAS, preprint 48.
- [BP97] A. D. Bryuno and V. I. Parusnikov. Comparison of various generalizations of continued fractions. *Mathematical notes*, 61:278–286, 1997. Keldysh Institute of the RAS, preprint 52.
- [Bre81] A. J. Brentjes. *Multi-dimensional Continued Fraction Algorithms*, volume 145 of *Mathematical Centre Tracts*. Mathematisch Centrum Amsterdam, 1981. MR 83b:10038.
- [Bri01] K. M. Briggs. On the Furtwängler algorithm for simultaneous rational approximation. *preprint*, ?:?, 2001.
- [Bri03] K. M. Briggs. Some explicit badly approximable pairs. *Journal of Number Theory*, 103:71–76, 2003. doi:10.1016/S0022-314X(03)00104-5.
- [Cla97] I. Vaughan L. Clarkson. *Approximation of Linear Forms by Lattice Points, with applications to signal processing*. PhD thesis, Australian National University, 1997.
- [Fur26] Ph. Furtwängler. Über die simultane Approximation von Irrationalzahlen (Erste Mitteilung). *Math. Annalen*, 96:169–175, 1926.

- [Fur28] Ph. Furtwängler. Über die simultane Approximation von Irrationalzahlen (Zweite Mitteilung). *Math. Annalen*, 99:71–83, 1928.
- [HK00a] D. M. Hardcastle and K. Khanin. Almost everywhere strong convergence of multidimensional continued fraction algorithms. Technical Report HPL-BRIMS-00-12, BRIMS, Bristol, UK, 2000.
- [HK00b] D. M. Hardcastle and K. Khanin. Continued fractions and the d -dimensional Gauss transform. Technical Report HPL-BRIMS-00-15, BRIMS, Bristol, UK, 2000.
- [IK02] M Iosifescu and C Kraaikamp. *Metrical Theory of Continued Fractions*. Kluwer, 2002.
- [Kara] O. N. Karpenkov. On constructing multidimensional periodic continued fractions. <http://front.math.ucdavis.edu/math.NT/0411031>.
- [Karb] O. N. Karpenkov. On examples of two-dimensional periodic continued fractions. <http://front.math.ucdavis.edu/math.NT/0411054>.
- [Kor94] E. Korkina. The periodicity of multidimensional continued fractions. *C. R. Acad. Sci.*, 319:777–780, 1994. MR 95j:11064.
- [Kor95] E. I. Korkina. Two-dimensional continued fractions. The simplest examples. *Proc. Steklov Institute of Mathematics*, 209:124–144, 1995. MR 97k:11104.
- [KS99] M. L. Kontsevich and Yu. M. Suhov. Statistics of Klein polyhedra and multidimensional continued fractions. In *Pseudoperiodic topology*, volume 197 of *Am. Math. Soc. Transl.*, pages 9–27. 1999. Dedicated to V. I. Arnold on his 60th anniversary.
- [Lac93] Gilles Lachaud. Polyèdre d’Arnol’d et voile d’un cône simplicial: analogues du théorème de Lagrange. *C. R. Acad. Sci. Paris*, 317:711–716, 1993.

- [Lac98a] Gilles Lachaud. Klein polygons and geometric diagrams. In *Contemporary mathematics*, volume 210, pages 365–372. 1998. MR 99a:11086.
- [Lac98b] Gilles Lachaud. Sails and Klein polyhedra. In *Contemporary mathematics*, volume 210, pages 373–385. 1998. MR 98k:11094.
- [LP95] J. C. Lagarias and Andrew D. Pollington. The continuous diophantine approximation mapping of Szekeres. *J. Austral. Math. Soc.*, A59:148–172, 1995.
- [Min11a] H. Minkowski. Zur geometrie der Zahlen. In David Hilbert, editor, *Gesammelte Abhandlungen*, volume 2, pages 43–52. 1911. reprinted Chelsea Pub. Co. 1967.
- [Min11b] H. Minkowski. Zur theorie der Kettenbrüche. In David Hilbert, editor, *Gesammelte Abhandlungen*, volume 1, pages 278–292. 1911. Reprinted Chelsea Pub. Co. 1967.
- [Mos99] N. G. Moshchevitin. Continued fractions, multidimensional Diophantine approximations and applications. *J. de Théorie des Nombres de Bordeaux*, 11:425–438, 1999. www.emis.de/journals/JTNB/.
- [RS92] A. M. Rockett and P. Szűsz. *Continued Fractions*. World Scientific, 1992.
- [Sch95] F. Schweiger. *Ergodic theory of fibred systems and metric number theory*. Clarendon Press, Oxford, 1995.
- [Sch96] W. W. Schmidt. *Diophantine Approximation*, volume 785 of *Lecture Notes in Mathematics*. Springer-Verlag, first edition, 1996. Second printing.
- [Sze70] G. Szekeres. Multidimensional continued fractions. *Annales Universitatis Scientiarum Budapestinenses de Rolando Eötvös Nominatate, sectio mathematica*, 8:113–140, 1970. MR 47 #1753.