Control of chaos by time delayed feedback

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... in collaboration with

Hartmut Benner
TU Darmstadt
Germany

Eckehard Schöll
TU Berlin
Germany

0 Introduction

chaos

$\Downarrow$

sensitivity &

large number of

unstable orbits

$\Downarrow$

control with

tiny forces

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control of chaos

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⇓

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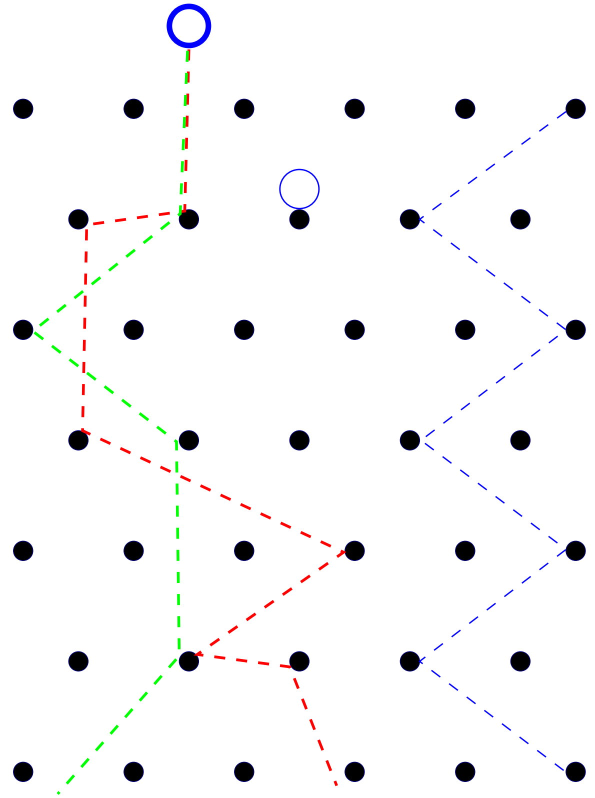
Galton board



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⇓

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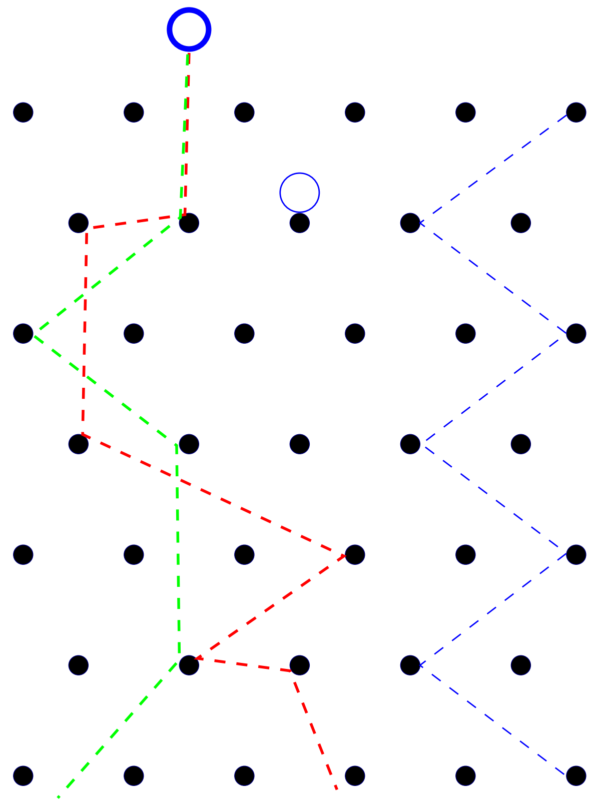

→ invasive vs. non invasive control schemes (cf. spectroscopy)

1 Time–delayed feedback method

Goal

stabilisation of
unstable periodic orbits

$$\xi(t) = \xi(t+T)$$

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Limitations

- no model
- no data processing

e.g.. fast time scales

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Idea (→ K. Pyragas, PLA **170**, 421, '92)

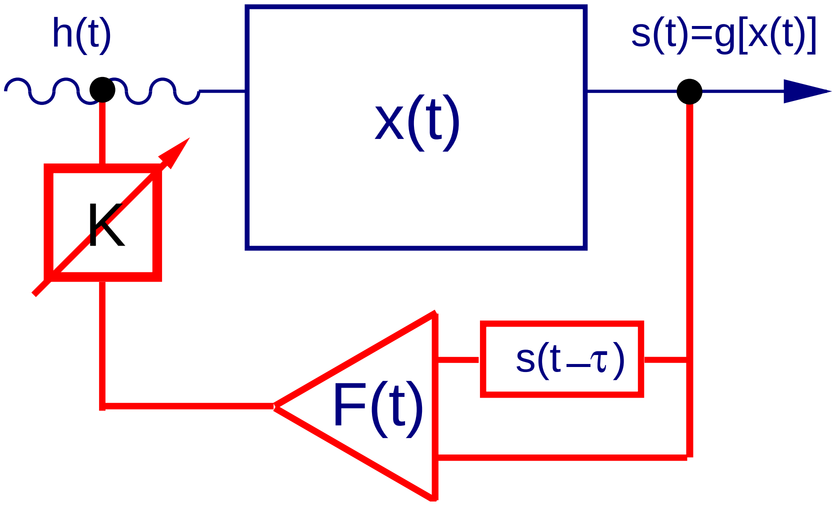
control force

$$F(t) \sim x(t) - \xi(t) \rightarrow x(t) - x(t-\tau)$$

delay $\tau \equiv$ period T

non invasive method

control scheme



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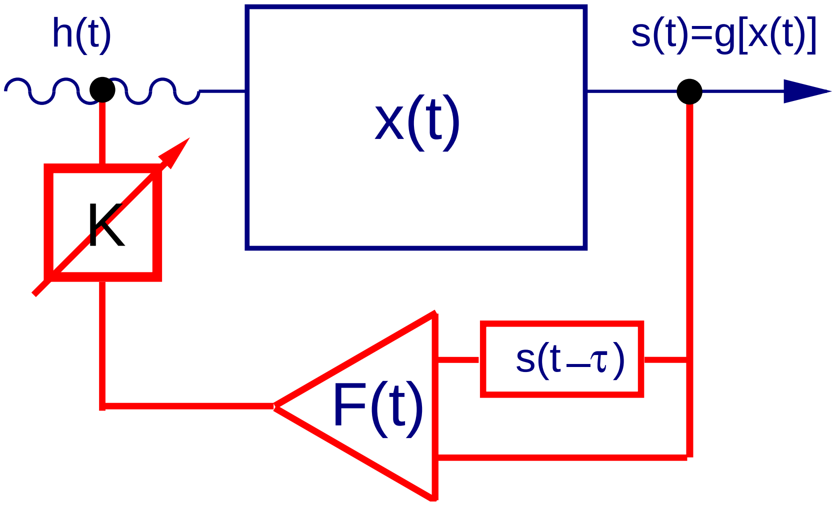
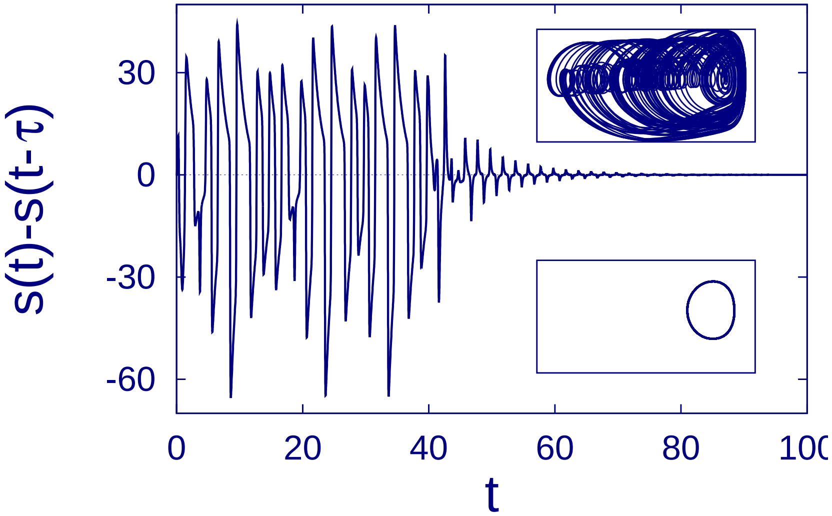


example



Experimental realisations

- CO_2 laser (→ S. Bielawski et.al., PRE **49**, R971, '94)
- discharge gas tube (→ T. Pierre et.al., PRL **76**, 2290, '96)
- nonlinear laser systems (→ E. Benkler et.al., PRL **84**, 879, '00)

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- FMR on YIG (→ H. Benner et.al., JKPS **40**, 1046, '02)
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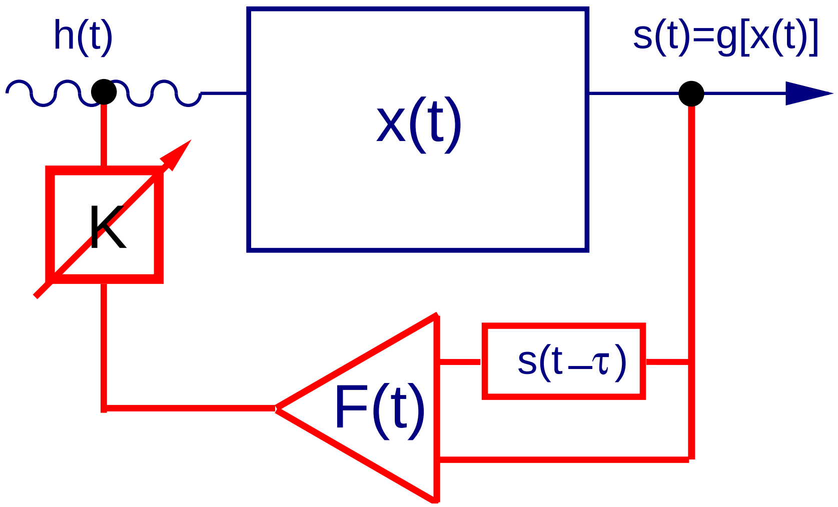
- electronic circuits (→ K. Pyragas et.al., PLA **180**, 99, '93)
- mechanical pendulum (→ T. Hikihara et.al. PLA **211**, 29, '96)

2 Stability analysis



2 Stability analysis

General equation of motion

$$\dot{x}(t) = f\left(x(t), F(t)\right)$$

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Control schemes

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$$F_{Pyr}(t) = K\left\{g[x(t)] - g[x(t-\tau)]\right\}$$

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- extended time–delayed feedback (→ J. E. S. Socolar et.al., PRE **50**, 3245, '94)

$$F_{ext}(t) = K \sum_{\nu \geq 0} R^{\nu} \left\{g[x(t-\nu\tau)] - g[x(t-\nu\tau-\tau)]\right\}$$

- rhythmic control (→ S. Bielawski et.el., PRA **47**, 2492, '93)

$$F_{rhy}(t) = K(t)\left\{g[x(t)] - g[x(t-\tau)]\right\}$$

Linear stability analysis: $x(t) = \xi(t) + \delta x(t)$

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$$\delta\dot{x}(t) = D_1 f(\xi(t),0)\delta x(t) + \underbrace{d_2 f(\xi(t),0) \otimes Dg[\xi(t)]}_{\text{control matrix}} K \left\{\delta x(t) - \delta x(t-\tau)\right\}$$

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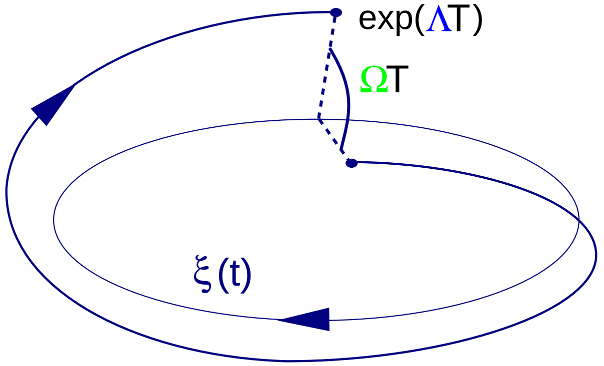
Floquet decomposition

$$\delta x(t) = e^{(\Lambda + i\Omega)t} Q(t)$$
$$Q(t) = Q(t+T)$$



Λ Expansion, Ω Torsion

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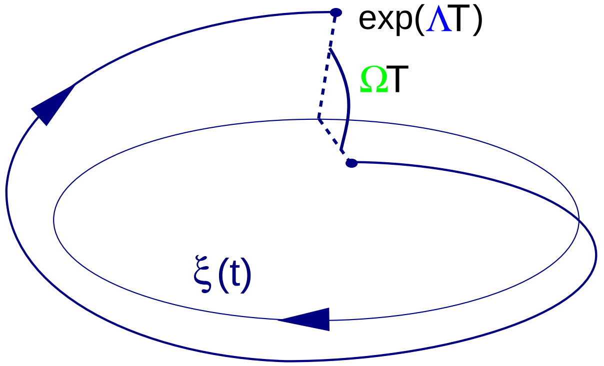

Λ Expansion, Ω Torsion

$$\delta x(t) - \delta x(t-\tau) \to \left\{1 - e^{-(\Lambda + i\Omega)\tau}\right\} Q(t)$$

Eigenvalue problem

$$(\Lambda + i\Omega)Q(t) + \dot{Q}(t) = (D_1 f(\xi(t), 0) + K \left\{ 1 - e^{-(\Lambda+i\Omega)\tau} \right\} d_2 f(\xi(t), 0) \otimes Dg[\xi(t)])Q(t)$$

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Characteristic equation

$$\Lambda + i\Omega = \Gamma\left[K\{1 - e^{-(\Lambda+i\Omega)\tau}\}\right]$$

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"Mean–field expansion"

$$\boxed{\Lambda\tau + i\Omega\tau = (\lambda + i\omega)\tau - (-\tau\chi)K\left\{1 - e^{-\Lambda\tau - i\Omega\tau}\right\}}$$

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- analytic expression (→ W.J. et.al., PRE **61**, 3675, '00)
- exact for diagonal control
- correct asymptotics for $K \to 0$ and $|K| \to \infty$

3 Control properties

- control domains and instabilities (→ W.J. et.al., PLA **254**, 158, '99)

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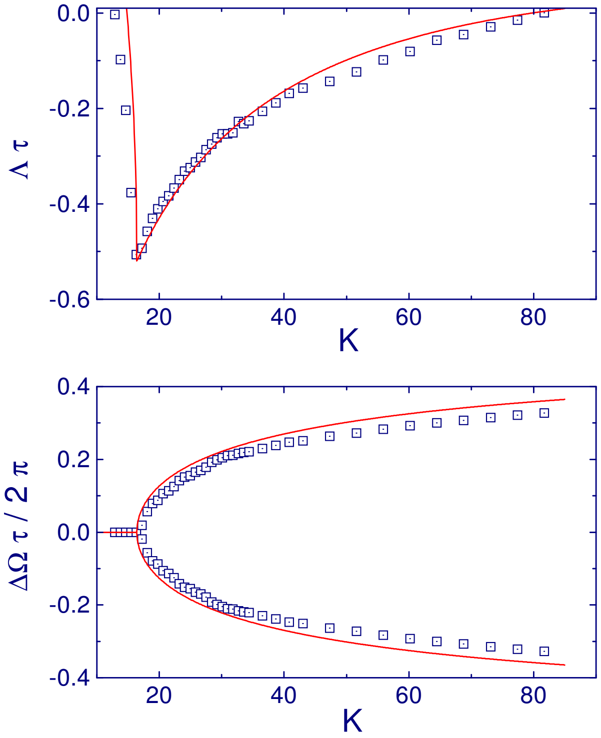
Floquet exponents



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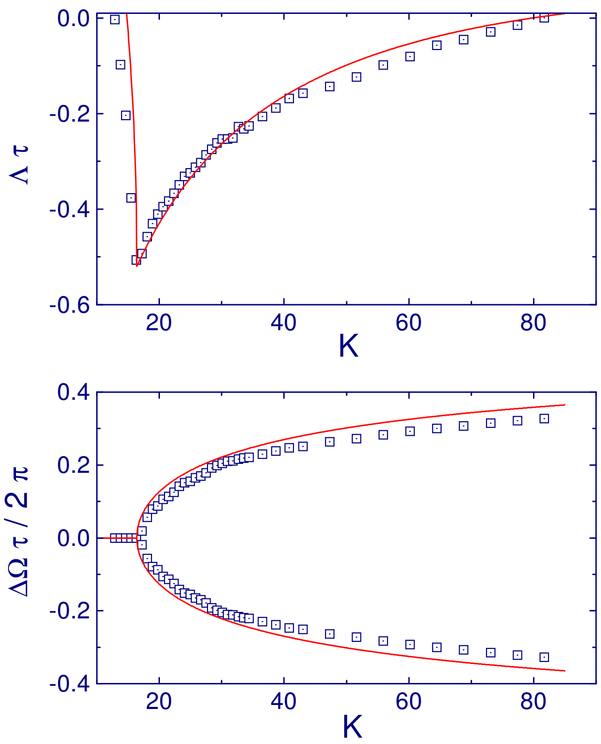
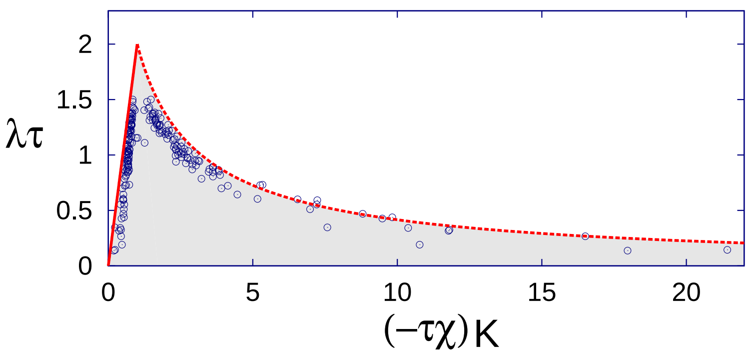


dependence on the Lyapunov exponent

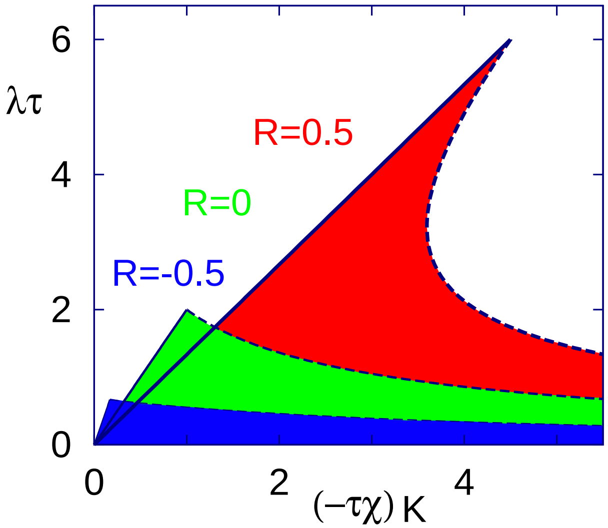


$\lambda\tau \lesssim 2$

– extended time–delayed feedback

– extended time–delayed feedback

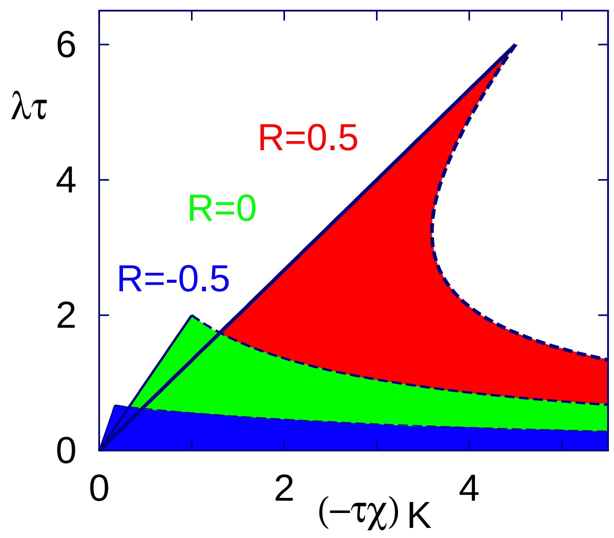
analytic expression



$$\lambda\tau \lesssim 2 \times \frac{1+R}{1-R}$$

– extended time–delayed feedback

analytic expression



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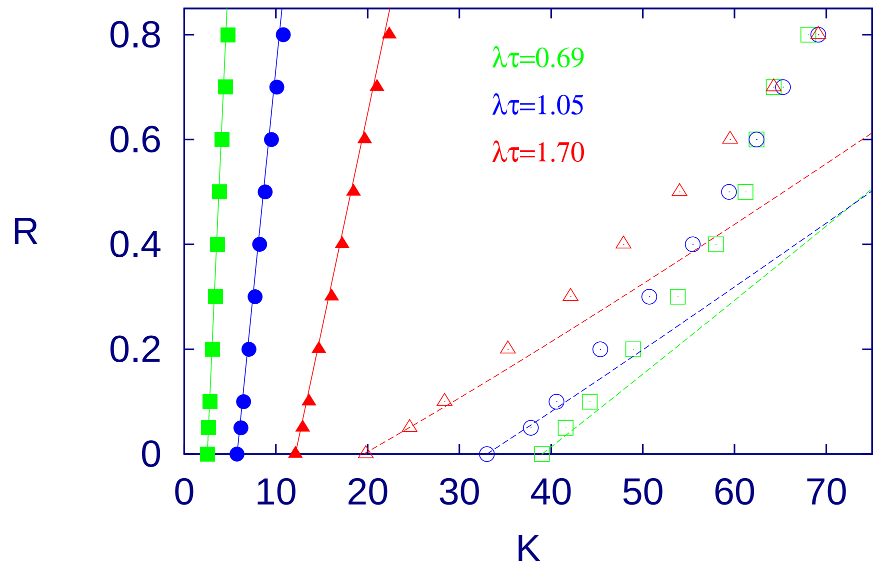
circuit experiment



- adjustment of delay (→ A. Kittel et.al., PLA **198**, 433, '95)

– induced periodic orbits (→ W.J. et.al., PRL **81**, 562, '98)

- topological constraints (torsion) (→ W.J. et.al., PRL **78**, 203, '97)

– rhythmic control, $K \longrightarrow K(t)$

– unstable controller (→ K.Pyragas, PRL **86**, 2265, '01)

- control loop latency, $F(t) \longrightarrow F(t-\delta)$ (→ W.J. et.al., PRE **59**, 2826, '99)

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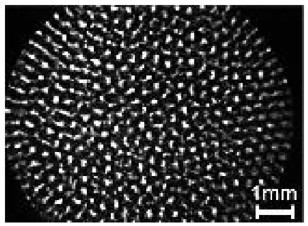
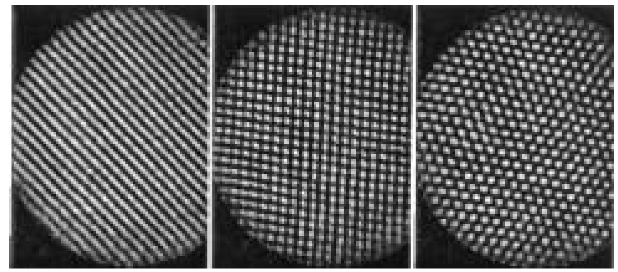
Limitations of the mean–field expansion (→ W.J. et.al., PRE **61**, 5045, '00)

4 Oscillating feedback and eigenmode control

Laser experiment: Fourier filter

(→ E. Benkler et.al., PRL **84**, 879, '00)





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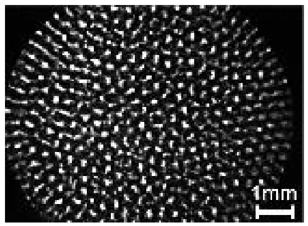

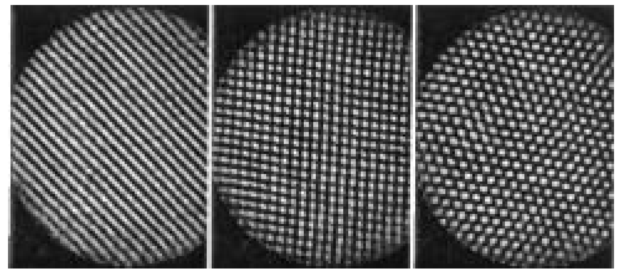

→ Control through eigenmodes

Simulation: Floquet mode control

(→ N. Baba et.al., PRL **89**, 074101, '02)

reaction–diffusion model

$$\partial_t a(x,t) = \frac{u-a}{1+(u-a)^2} - Ta + \partial_x^2 a - f_a(x,t)$$

$$\partial_t u(t) = \alpha \left[j_0 - \left(u - \int_0^L a dx/L \right) \right] - f_u(t)$$

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control force through spatio–temporal filters ...

$$f_a(x,t) = K \Psi_a(x,t) [s(t) - s(t-\tau)]$$

$$f_u(t) = K \Psi_u(t) [s(t) - s(t-\tau)]$$

$$s(t) = \int_0^L \Phi_a^*(x,t) a(x,t)\, dx + \Phi_u^*(t) u(t)$$

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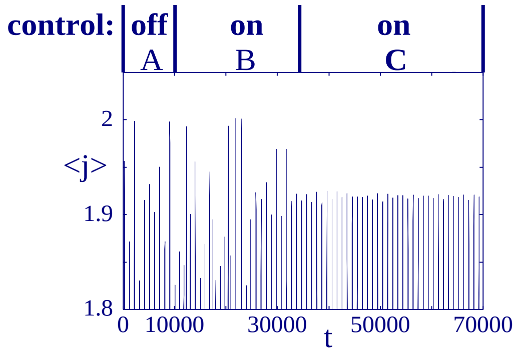
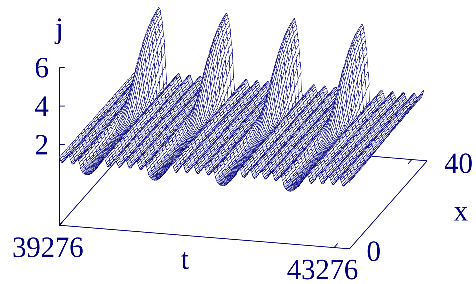
... derived from eigenmodes (Ψ_a, Ψ_u), (Φ_a, Φ_u) of the unstable orbit (→ analytic treatment).

stabilisation of filaments





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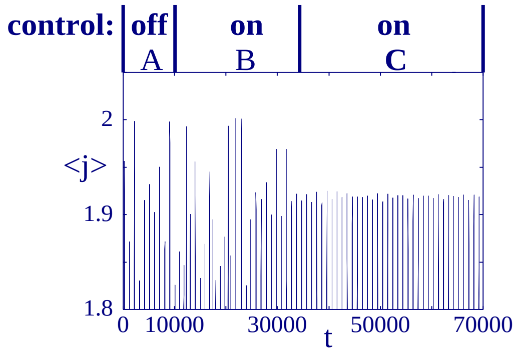
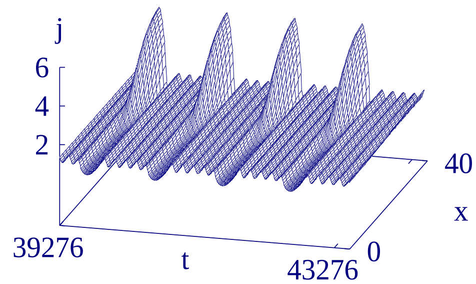
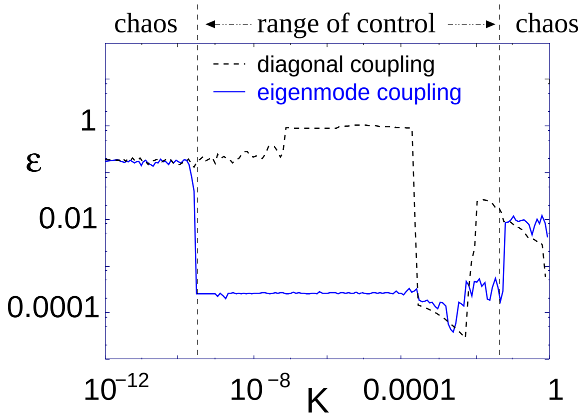




increase of control domain



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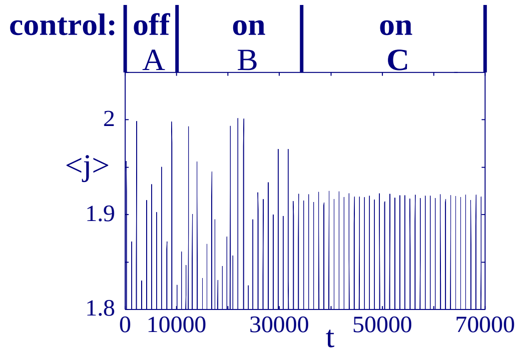
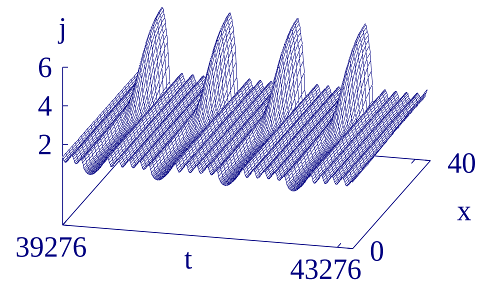
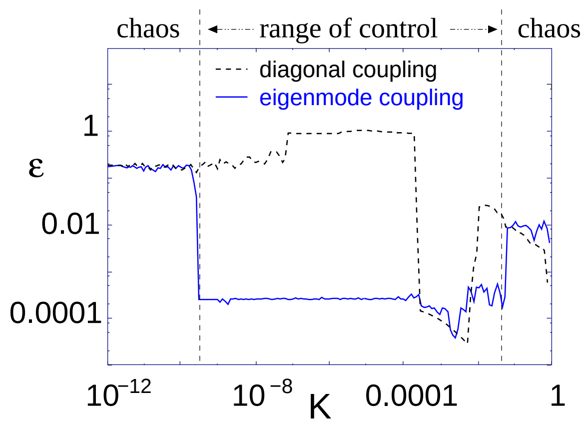
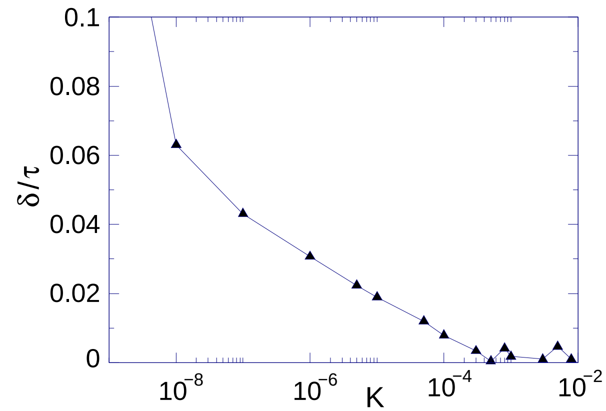




increase of control domain ... through phase shift δ.





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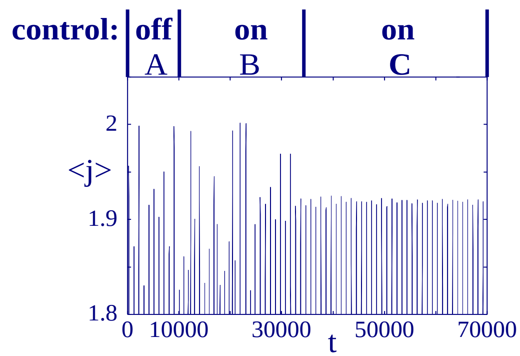

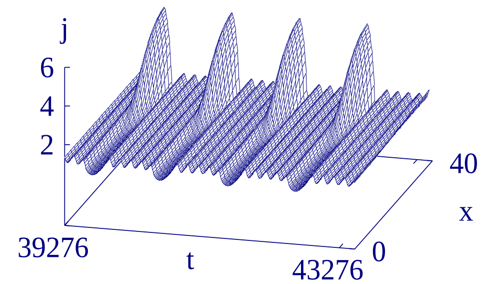

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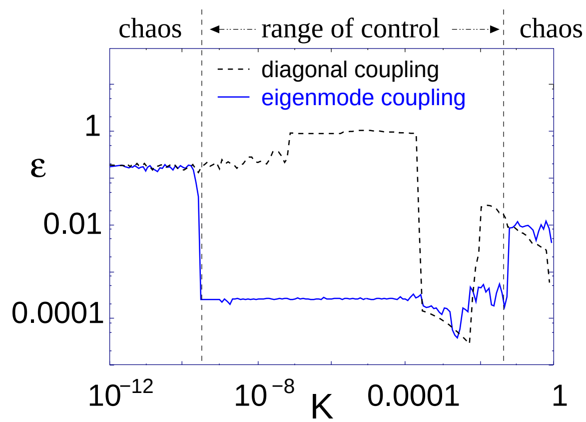

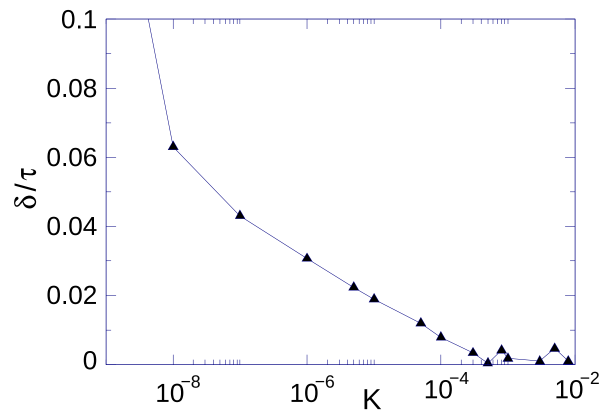

analytic prediction of the control interval (→ W. J. et.al., PRE **67**,026222, '03)

$$K_{max}/K_{min} \simeq \exp(A\delta^2)$$

Analytic example: (→ A. Fichtner, thesis TU Chemnitz, '03)
logistic map with oscillating feedback $x_{n+1} = f(x_n) + K_n(x_n - x_{n-p})$

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selection of the phase $(p = 2)$:

$$K_0 \quad K_1 \quad K_0 \quad K_1 \quad K_0 \quad K_1 \quad \ldots$$
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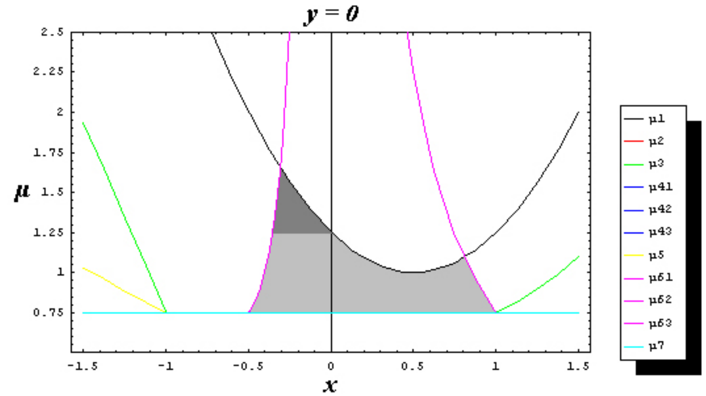
\end{array}
$$

without oscillation ($K_0 = K_1$)



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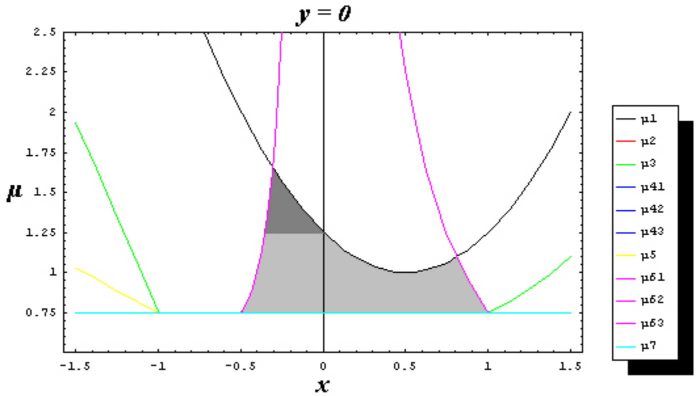
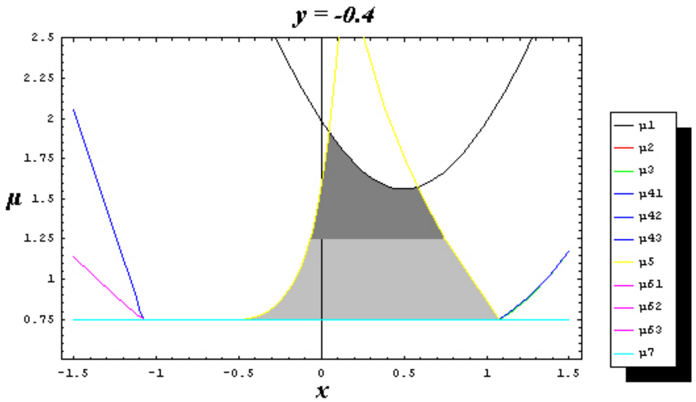
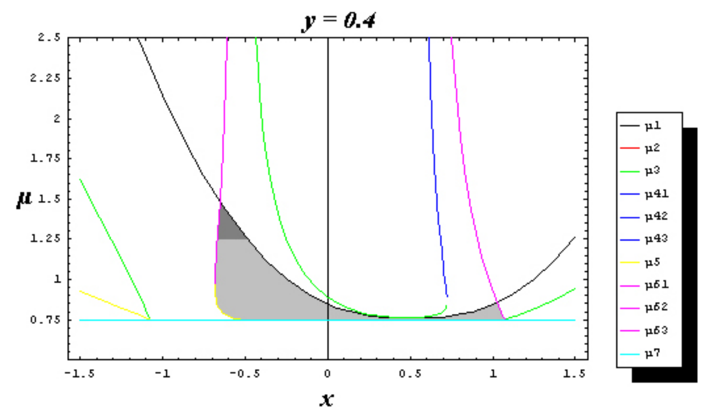


with oscillation of amplitude $y = (K_1 - K_0)/2$





5 From local to global analysis ?

- local analysis $\longrightarrow$ linear stability
- global features $\longrightarrow$ domain of attraction ?
- delay systems $\longrightarrow$ infinite–dimensional phase spaces

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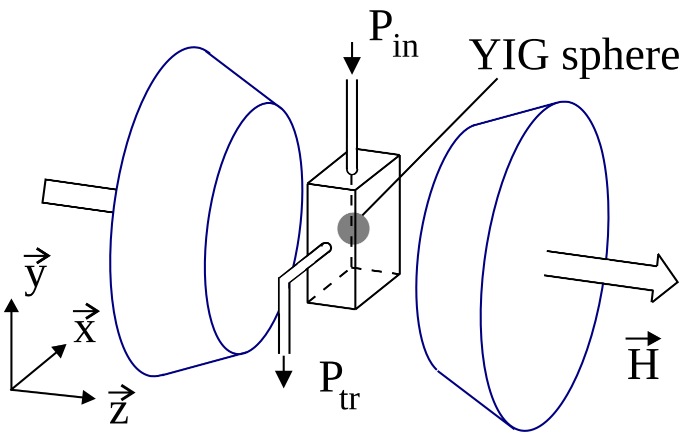
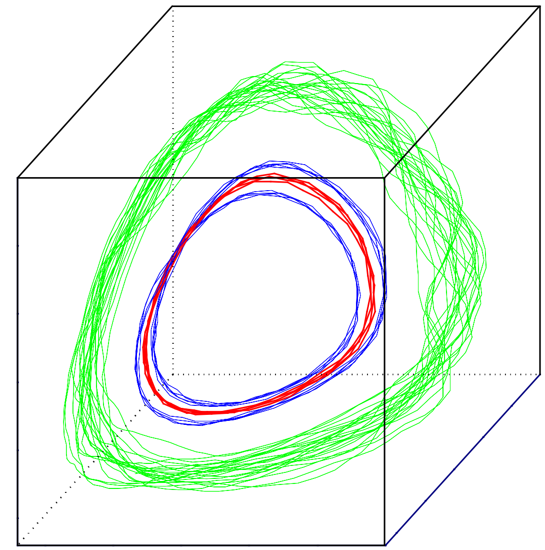
super/sub–critical instabilities

$\longrightarrow$ normal form analysis, amplitude equations, etc.

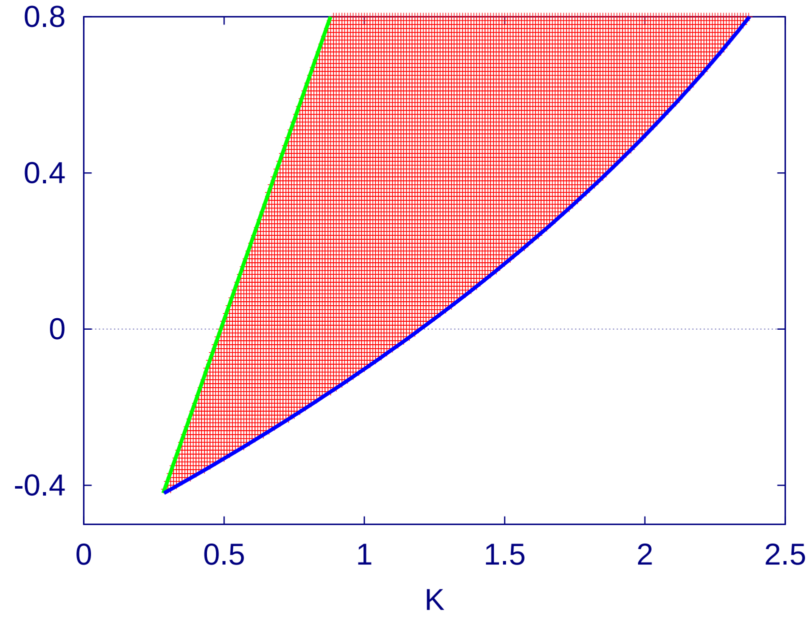
$$\dot{z} = \mu z - r|z|^2 z, \quad \text{sign}(\text{Re}(r)) = ?$$

Experimental hint (FMR in YIG)





Theoretical analysis (for extended control scheme)

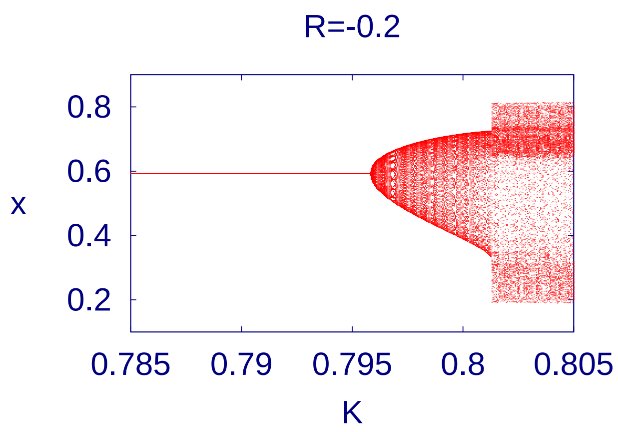


Flip: $K_{fl}/(1+R) = \text{const.}$, $r = \text{const.}$

Hopf: super/sub–critical transition possible

Simulation (Henon map with extended control)

upper control threshold K_{ho}



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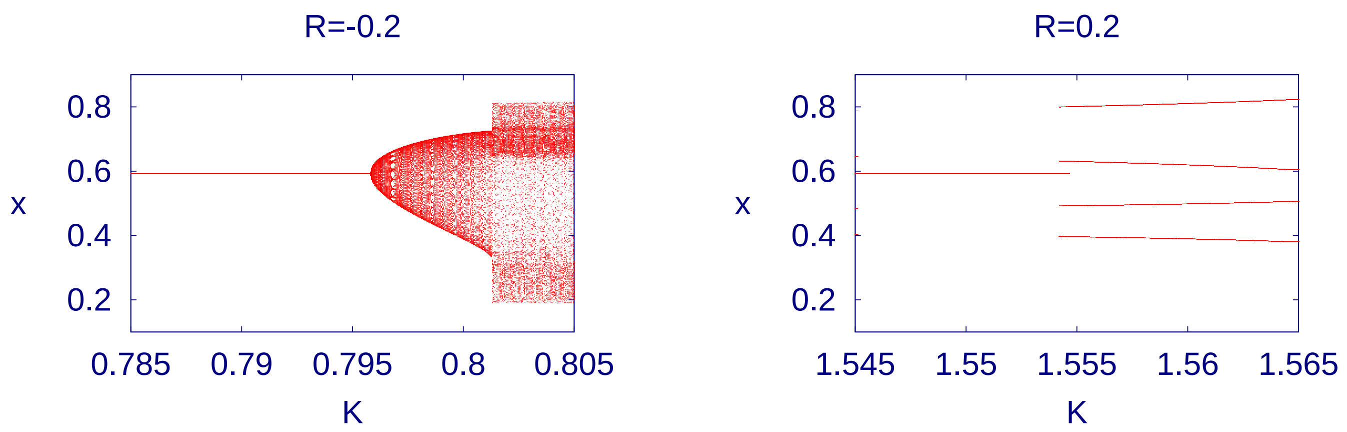

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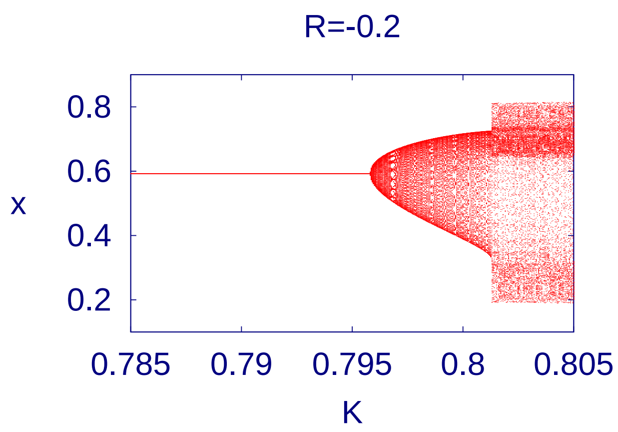
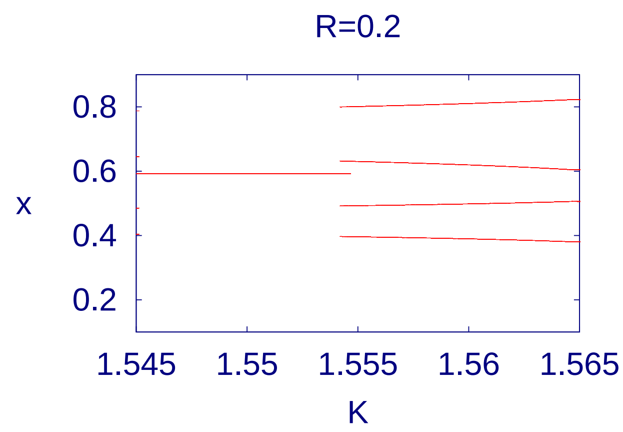
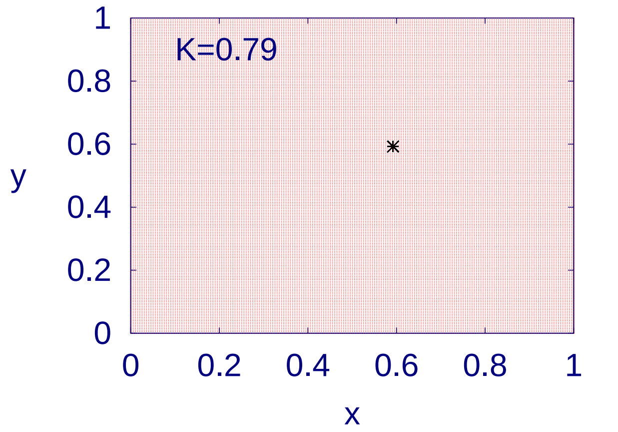




basin of attraction ($F_0 = 0$)



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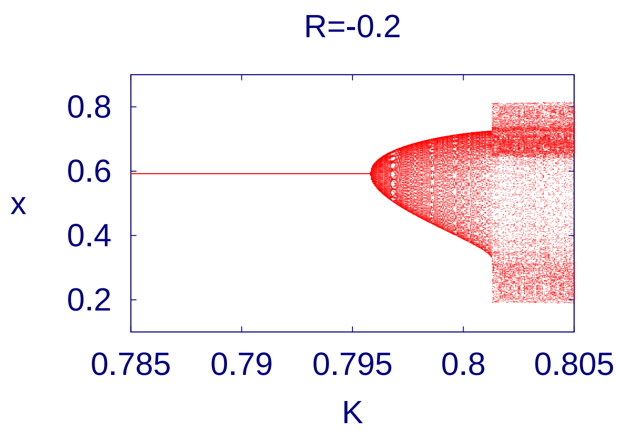

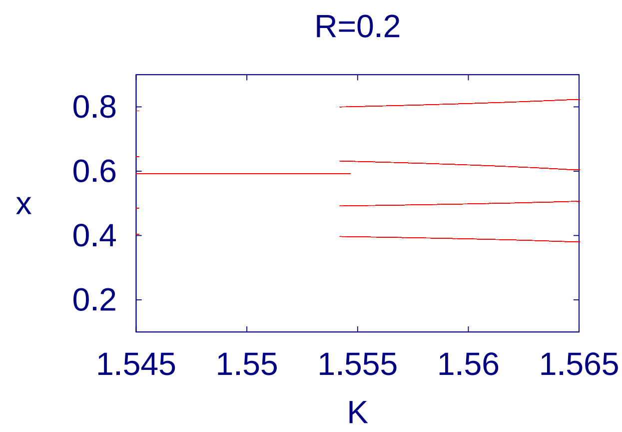

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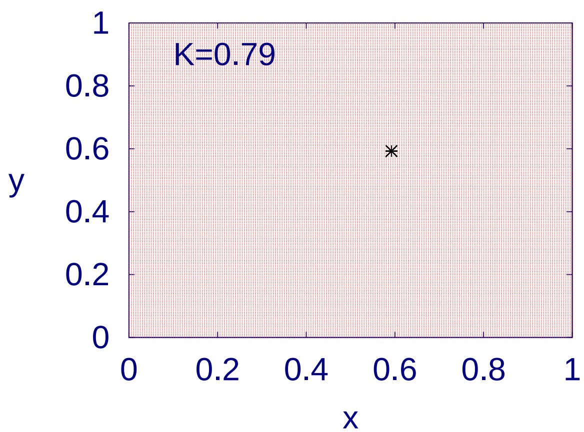

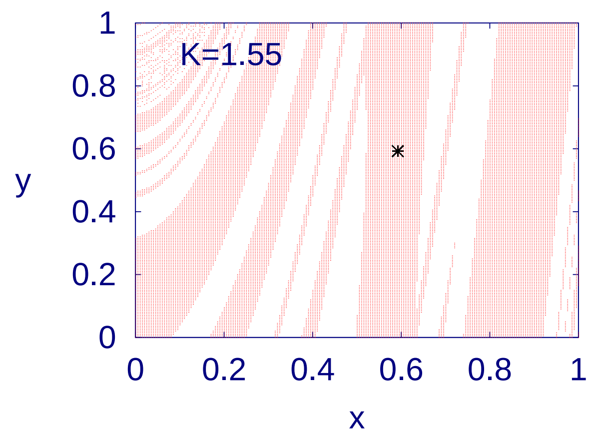

6 Outlook

– spatially extended systems
spatio–temporal delay, pattern selection, transport, . . .

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– spatially extended systems
  spatio–temporal delay, pattern selection, transport, . . .

– global properties of delay systems
  beyond (linear) stability analysis, manifolds, dimensions, visualisation, . . .

– noise & delay
  tunnelling with delay (→ Tsimring et.al., PRL **87**, 250602, '01)
  time scales, synchronisation, . . .