Problem 11

a) \( |T'(x)| = 2 \). Suppose \((x_0, x_1, \ldots, x_{p-1})\) is an orbit of period \(p\). Then

\[
|T'(x_p)| = |T'(x_{p-1})| |T'(x_{p-2})| \cdots |T'(x_0)| = 2^p > 1
\]

i.e., any periodic orbit is repelling (to make the reasoning waterproof one has to exclude that \(x = 1/2\) is a point of a periodic orbit. But that is rather obvious as \(T^{(k)}(1/2) = 0\) for any \(k \geq 2\)).

b) Since \(F_4\) is conjugate to \(T\), any periodic orbit of \(F_4\) is repelling as well.

Problem 12

a) \( f'(0) = 3 > 1 \Rightarrow x_0 = 0 \) repelling.

b) (use \(D_n = f(D_{n+1})\) and \(D_n \subseteq D_{n+1}\)).

\[
\begin{align*}
D_0 & = \{0\} \\
D_1 & = \{0, 1\} \quad (\text{since } f(1) = 0) \\
D_2 & = \{0, 1\} \cup [1/3, 2/3] \quad (\text{since } f([1/3, 2/3]) = \{1\}) \\
D_3 & = \{0, 1\} \cup [1/3, 2/3] \cup [1/9, 2/9] \cup [7/9, 8/9] \quad (\text{since } f([1/9, 2/9]) = f([7/9, 8/9]) = [1/3, 2/3]) \\
& \vdots
\end{align*}
\]

\[
\begin{align*}
D_0 & = \{0\} \\
D_1 & = \{0, 1\} \\
D_2 & = [1/3, 2/3] \\
D_3 & = [1/9, 2/9] \cup [7/9, 8/9] \\
& \vdots
\end{align*}
\]

cf. middle third Cantor set construction.

c) \[
\begin{align*}
|S_0| & = |(0, 1)| = 1 \\
|S_1| & = |(0, 1)| = 1 \\
|S_2| & = |(0, 1/3) \cup (2/3, 1)| = 2 \cdot \frac{1}{3} \\
|S_3| & = |(0, 1/9) \cup (2/9, 3/9) \cup (6/9, 7/9) \cup (8/9, 9/9)| = 4 \cdot \frac{1}{9} = \left(\frac{2}{3}\right)^2 \\
|S_4| & = 8 \cdot \frac{1}{27} = \left(\frac{2}{3}\right)^3 \\
& \vdots
\end{align*}
\]
i.e., $|S_n| = (2/3)^{n-1}$ since $S_n$ consists of $2^{n-1}$ open intervals of length $1/3^{n-1}$ each (follows by induction using $f(S_n) = S_{n-1}$, $S_n \subseteq [0, 1/3] \cup [2/3, 1]$ for any $n \geq 2$, and $f$ being linear with slope $\pm 3$ on $[0, 1/3]$ and on $[2/3, 1]$).

Thus $|S_\infty| = 0$, i.e., (“Lebesgue”) almost all initial conditions are mapped in a finite number of steps to the unstable fixed point $x_* = 0$. 