Problem 3

a) \[ f'_A(x) = 6x(1-x) \geq 0 \] monotonic increasing \( \Rightarrow \) each \( \omega \)-limit set contains a single fixed point.

\[ f'_B(x) = -2x \leq 0 \] monotonic decreasing \( \Rightarrow \) each \( \omega \)-limit set contains a period two orbit (i.e. either a single fixed point or two points of a proper period two).

\[ f'_C(x) = -(2 - x)^{-2}, \ |f'_C(x)| < 1 \] if \( x \in (0,1) \) \( \Rightarrow \) unique fixed point in \( (0,1) \). There is only one \( \omega \)-limit set which contains a single point.

b) \[ x_* = f_A(x_*) = 3x_*^2 - 2x_*^3 \Rightarrow x_*(2x_*^2 - 3x_* + 1) = 0. \] Thus \( x_* = 0 \) or \( x_* = 1 \) or \( x_* = 1/2 \).

- fixed point(s): \( x_* = f_B(x_*) = 1 - x_*^2 \Rightarrow x_*^2 + x_* - 1 = 0 \)

\[ x_* = -1/2 + \sqrt{1/4 + 1} = \frac{\sqrt{5} - 1}{2} \]

proper period two orbit: \( x_0 = 1 - x_1^2, \ x_1 = 1 - x_0^2 \Rightarrow x_0 - x_1 = (x_0 - x_1)(x_0 + x_1) \Rightarrow 1 = x_1 + x_0 \) (if \( x_0 \neq x_1 \)). Thus \( x_0 = 0 \) and \( x_1 = 1 \) (or vice versa).

- fixed point: \( x_* = f_C(x_*) = (1 - x_*)/(2 - x_*) \Rightarrow x_*^2 - 3x_* + 1 = 0 \). Thus

\[ x_* = 3/2 - \sqrt{9/4 - 1} = \frac{3 - \sqrt{5}}{2} \]
Problem 4

$f$ continuous and invertible implies $f$ monotonic

(otherwise: there exists $x_1 < x_2 < x_3$ such that $f(x_1) < f(x_2) > f(x_3)$, or $f(x_1) > f(x_2) < f(x_3)$. Choose $y$ such that $f(x_1) < y < f(x_2) > y > f(x_3)$. The intermediate value theorem guarantees that there are $\xi_1$ and $\xi_2$, $x_1 < \xi < x_2$ and $x_2 > \xi > x_3$ such that $f(\xi_1) = f(\xi_2) = y \Rightarrow$.)

Thus each $\omega$-limit set contains a fixed point.

- The conclusion does not hold for discontinuous invertible maps, e.g.,

\[
f(x) = \begin{cases} 
  2/3 + x & \text{if } 0 \leq x < 1/3 \\
  x - 1/3 & \text{if } 1/3 \leq x < 1 \\
  1 & \text{if } x = 1 
\end{cases}
\]

$x_0 = 1/6$, $x_1 = 5/6$, $x_2 = 1/2$, proper period three orbit.

- The conclusion does not hold for open intervals e.g.

\[f(x) = x^2\]

No fixed point in $(0, 1)$. 