Problem 1

a)

\[ \dot{x}(t) = -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t) = v(t) \]
\[ \dot{v}(t) = -\omega^2 x_0 \cos(\omega t) - \omega v_0 \sin(\omega t) = -\omega^2 x(t) \]

b)

\[ \Phi_s(\Phi_t(x_0, v_0)) = \left( x(t) \cos(\omega s) + \frac{v(t)}{\omega} \sin(\omega s), -\omega x(t) \sin(\omega s) + v(t) \cos(\omega s) \right) \]
\[ = \left( x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \right) \cos(\omega s) + \left\{ -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t) \right\} \cos(\omega s), \]
\[ -\omega \left\{ x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \right\} \sin(\omega s) + \left\{ -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t) \right\} \sin(\omega s) \]
\[ = \left( x_0 \cos(\omega (t + s)) + \frac{v_0}{\omega} \sin(\omega (t + s)), -\omega x_0 \sin(\omega (t + s)) + v_0 \cos(\omega (t + s)) \right) \]
\[ = \Phi_{t+s}(x_0, v_0) \]

Problem 2

a) For instance: \( r^2 = x^2 + y^2 \), \( x = r \cos \varphi \), \( y = r \sin \varphi \)

\[ r\dot{r} = x\dot{x} + v\dot{v} = x(v - x) + v(-x - v) = -r^2 \quad \Rightarrow \quad \dot{r} = -r \quad \Rightarrow \quad r(t) = r(0)e^{-t} \]
\[ r^2\dot{\varphi} = -\dot{x}v + \dot{v}x = -(v - x)v + (-x - v)x = -r^2 \quad \Rightarrow \quad \dot{\varphi} = -1 \quad \Rightarrow \quad \varphi(t) = \varphi(0) - t \]

Thus

\[ x(t) = r(0)e^{-t} \cos(\varphi(0) - t) \]
\[ = r(0) \cos(\varphi(0))e^{-t} \cos(t) + r(0) \sin(\varphi(0))e^{-t} \sin(t) \]
\[ = x_0 e^{-t} \cos(t) + v_0 e^{-t} \sin(t) \]
\[ v(t) = r(0)e^{-t} \sin(\varphi(0) - t) \]
\[ = r(0) \sin(\varphi(0))e^{-t} \cos(t) - r(0) \cos(\varphi(0))e^{-t} \sin(t) \]
\[ = v_0 e^{-t} \cos(t) - x_0 e^{-t} \sin(t) \]
\[ \Phi_t(x_0, v_0) = (x_0 e^{-t} \cos(t) + v_0 e^{-t} \sin(t), v_0 e^{-t} \cos(t) - x_0 e^{-t} \sin(t)) \]

b)

\[ t = 0 \quad : \quad x_0 = 0, \quad v_0 \geq 0 \]
\[ x(t) = v_0 e^{-t} \sin(t), \quad v(t) = v_0 e^{-t} \cos(t) \]
\[ t = T \quad : \quad x(T) = v_0 e^{-T} \sin(T) = 0, \quad v(T) = v_0 e^{-T} \cos(T) \geq 0 \quad \Rightarrow \quad T = 2\pi \]
\[ v_1 = v(T) = v_0 e^{-T} \]

Thus

\[ P : \Sigma \rightarrow \Sigma, \quad v_n \mapsto v_{n+1} = v_n e^{-2\pi} \]