i) \( \int_I \rho(x) \, dx = 1 \), and

ii) \( \int_I h(x) \rho(x) \, dx = \int_I h(f(x)) \rho(x) \, dx \) for any (integrable) function \( h : I \to \mathbb{R} \).

How to obtain an equation for \( \rho \)? Use substitution.

For simplicity assume that \( f \) has two invertible branches, \( f_0 : I_0 \to I \), \( f_1 : I_1 \to I \), i.e.,

\[
\begin{align*}
   f_0^{-1} & : I \to I_0 \\
   f_1^{-1} & : I \to I_1
\end{align*}
\]

Then

\[
\int_I h(f(x)) \rho(x) \, dx = \int_{I_0} h(f_0(x)) \rho(x) \, dx + \int_{I_1} h(f_1(x)) \rho(x) \, dx
\]

\[
x = f_0^{-1}(y), \quad dx = |(f_0^{-1})'(y)| \, dy \quad x = f_1^{-1}(y), \quad dx = |(f_1^{-1})'(y)| \, dy
\]

\[
= \int_I h(y) \rho(f_0^{-1}(y)) \frac{dy}{|f_0'(f_0^{-1}(y))|} + \int_I h(y) \rho(f_1^{-1}(y)) \frac{dy}{|f_1'(f_1^{-1}(y))|}
\]

\[
= \int_I h(y) \rho(y) \, dy.
\]

Thus

\[
\rho(x) = \sum_{i=0}^{1} \frac{1}{|f'(f_i^{-1}(x))|} \rho(f_i^{-1}(x)) = \sum_{y \in f_i^{-1}(x)} \frac{1}{|f'(y)|} \rho(y)
\]

and we have proven the proposition:

**Proposition 3.1** Let \( f : I \to I \) denote a (piecewise) smooth map with invariant density \( \rho \). Then the density obeys the so called Frobenius-Perron equation

\[
\rho(x) = \sum_{y \in f^{-1}(x)} \frac{1}{|f'(y)|} \rho(y)
\]

**Example 3.1** Tent map \( T(x) = 1 - |2x - 1| \)
\( T^{-1}(x) = \{ \frac{x}{2}, 1 - \frac{x}{2} \}, \ |T'(x)| = 2. \)

Frobenius-Perron equation
\[
\rho(x) = \frac{1}{2} \rho\left(\frac{x}{2}\right) + \frac{1}{2} \rho\left(1 - \frac{x}{2}\right)
\]

\( \rho(x) = 1 \) is an invariant density of the tent map since \( \int_0^1 1 \, dx = 1 \) and
\[
\frac{1}{2} \rho\left(\frac{x}{2}\right) + \frac{1}{2} \rho\left(1 - \frac{x}{2}\right) = \frac{1}{2} + \frac{1}{2} = \rho(x)
\]

That means orbit points are uniformly distributed.

**Remark:** \( \rho(x) \) may be considered as a dynamically invariant probability distribution:

- \( \rho(x) \, dx \) probability to find a point in \((x, x + dx)\).
- \( \rho\left(\frac{x}{2}\right) \, \frac{dx}{2} + \rho\left(1 - \frac{x}{2}\right) \, \frac{dx}{2} \) probability to find a point to be mapped into \((x, x + dx)\).
- Invariance condition
\[
\rho(x) \, dx = \rho\left(\frac{x}{2}\right) \, \frac{dx}{2} + \rho\left(1 - \frac{x}{2}\right) \, \frac{dx}{2}
\]

\( \rho(x) \) is easy to compute for the tent map. How to generalise to a larger class of systems?

**b) Piecewise linear Markov maps**

**Definition 3.2** An expansive Markov map \( f : I \to I \) with Markov partition \( \{I_0, \ldots, I_{N-1}\} \) is said to be piecewise linear if \( f(x) \) has constant slope on \( \text{int}(I_k) \), i.e., \( f'(x) = \gamma_k \) for \( x \in \text{int}(I_k) \), \( k = 0, \ldots, N - 1 \).
Example 3.2 Expansive Markov maps

- (see example 2.3)

\[ I_0 = \left[ 0, \frac{1}{2} \right], \quad \gamma_0 = -2 \]

\[ I_1 = \left[ \frac{1}{2}, 1 \right], \quad \gamma_1 = 1 \]

piecewise linear Markov map.

- \( f(x) = \frac{6x}{1 + x} \mod 1 \) on \([0, 1)\).

Markov partition

\[ I_0 = \left[ 0, \frac{1}{5} \right], I_1 = \left[ \frac{1}{5}, \frac{1}{2} \right], I_2 = \left[ \frac{1}{2}, 1 \right) \]

expansive \( |f'(x)| = \left| \frac{6}{(1 + x)^2} \right| \geq \frac{3}{2} \), but not piecewise linear.

Proposition 3.2 Let \( f \) be a piecewise linear Markov map with Markov partition \( \{I_0, \ldots, I_{N-1}\} \), slopes \( \gamma_k = f'(x)|_{x \in \text{int}(I_k)} \), and transition matrix \( A \). Then an invariant density can be written as

\[ \rho(x) = \sum_{k=0}^{N-1} \rho_k \chi_k(x) \]

where \( \chi_k \) denotes the characteristic function of \( I_k \) (i.e., \( \chi_k(x) = 1 \) if \( x \in I_k \), 0 otherwise) and the vector \( (\rho_0, \rho_1, \ldots, \rho_{N-1}) \) is eigenvector of the so called transfer matrix

\[ T_{k\ell} = \frac{A_{k\ell}}{|\gamma_\ell|} \]

with eigenvalue one (i.e., \( \rho_k = \sum_{\ell=0}^{N-1} \frac{A_{k\ell}}{|\gamma_\ell|} \rho_\ell \)).

Proof: