Example 2.4 $f : [0, 1] \to [0, 1]$

$I_0 = [0, \frac{1}{2}], I_1 = [\frac{1}{2}, 1]$ is not a Markov partition since $f(\text{int}(I_1)) = (0, \frac{3}{4})$, i.e., $f(\text{int}(I_1)) \cap \text{int}(I_1) = (\frac{1}{2}, \frac{3}{4}) \neq \emptyset$, but $\text{int}(I_1)$ is not contained in $f(\text{int}(I_1))$.

Is there another partition which is a Markov partition?

Markov maps map boundary points of the partition on boundary points.

**Definition 2.4** The $N \times N$ matrix defined by

$$A_{k\ell} = \begin{cases} 1 & \text{if } f(\text{int}(I_k)) \supseteq \text{int}(I_\ell) \\ 0 & \text{if } f(\text{int}(I_k)) \cap \text{int}(I_\ell) = \emptyset \end{cases}$$

is called **topological transition matrix** of the Markov map $f$ ("$A_{k\ell} = 1$ if the transition $I_k \to I_\ell$ is permitted")

**Example 2.5** (see examples 2.2 and 2.3)

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
Definition 2.5 A Markov map is said to be an expansive (expanding) Markov map if $f$ is smooth on each $\text{int}(I_k)$ and $|f'(x)| \geq \lambda > 1$ for $x \in \text{int}(I_k)$, $k = 0, \ldots, N - 1$.

Remark: It is sufficient to require expansiveness for some iterate, say $|(f^{(n)})'(x)| \geq \lambda > 1$.

Example 2.6 (see example 2.5)

\begin{itemize}
  \item $T(x) = 1 - |2x - 1|$ expansive since $|T'(x)| = 2 > 1$ if $x \in (0, \frac{1}{2})$ or $x \in (\frac{1}{2}, 1)$.
\end{itemize}

\begin{itemize}
  \item $|f'(x)| = 2$ if $x \in \text{int}(I_0)$ but $|f'(x)| = 1$ if $x \in \text{int}(I_1)$. Nevertheless, $|(f^{(2)})'(x)| = |f'(f(x))||f'(x)| = 2 \cdot 1$ if $x \in \text{int}(I_1)$, as $x \in I_1$ implies $f(x) \in I_0$, i.e., $f'(x) = 1$ and $f'(f(x)) = -2$. In addition $|(f^{(2)})'(x)| \geq 2$ if $x \in \text{int}(I_0)$. Thus the map is still expansive.
\end{itemize}

c) Symbolic dynamics
Main idea: Consider a Markov map $f$ with Markov partition $\{I_0, I_1, \ldots, I_{N-1}\}$ and symbols $\sigma \in \{0, 1, \ldots, N-1\}$. Assign to $x_0 \in I$ the symbol sequence $\sigma_0 \sigma_1 \sigma_2 \ldots$ according to the rule $x_k = f^{(k)}(x_0) \in I_{\sigma_k}$. Then $x_n$ has symbol sequence $\sigma_n \sigma_{n+1} \ldots$.

**Definition 2.6** Let $f$ be a Markov map with transition matrix $A$. A symbol sequence $\sigma_0 \sigma_1 \sigma_2 \ldots$ is called **admissible** if $A_{\sigma_k \sigma_{k+1}} = 1$, $k \geq 0$.

**Example 2.7** Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

- $0.0110010\ldots = \sigma_0 \sigma_1 \sigma_2 \ldots$ is not admissible since $A_{\sigma_1 \sigma_2} = A_{11} = 0$.
- $0.100010\ldots$ is admissible.

All symbol sequences which contain a pair “11” are not admissible since $A_{11} = 0$.

**Proposition 2.1** Let $f$ be a continuous expansive Markov map with transition matrix $A$. For any admissible symbol sequence $\sigma_0 \sigma_1 \sigma_2 \ldots$ there exists a unique point $x \in I$ such that $f^{(k)}(x) \in I_{\sigma_k}$ (for any $k \geq 0$).

**Proof:** Main idea: tent map. Fix a symbol sequence, say $0.101\ldots$

- $I_1$: $x$-values such that $x \in I_1$, $|I_1| = \frac{1}{2}$.
- $I_{10} = I_1 \cap T^{-1}(I_0)$: $x$-values such that $x \in I_1$ and $T(x) \in I_0$, $|I_{10}| = \frac{1}{2} \cdot \frac{1}{2}$. $I_{10}$ is a closed interval, $I_{10} \subseteq I_1$ and $T : I_{10} \to I_0$ is bijective.
- $I_{101} = I_{10} \cap T^{-2}(I_1) = I_1 \cap T^{-1}(I_{10})$: $x$-values such that $x \in I_1$, $T(x) \in I_0$, and $T^2(x) \in I_1$, $|I_{101}| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$. $I_{101}$ is a closed interval, $I_{101} \subseteq I_{10}$ and $T : I_{101} \to I_{101}$ is bijective.