• $x_\ast = 1$ is limit point of $(0, 1)$ since $(1 - \varepsilon, 1 + \varepsilon) \cap (0, 1) = (1 - \varepsilon, 1) \neq \emptyset$ for any $\varepsilon > 0$. $x_\ast > 1$ is not a limit point as $(0, 1) \cap U_\varepsilon(x) = \emptyset$ if \varepsilon < $x_\ast - 1$.

• $\text{cl}((0, 1)) = [0, 1]$ as $x_\ast = 0$, $x_\ast = 1$, and any $x_\ast \in (0, 1)$ are limit points. $[0, 1]$ is closed since $[0, 1] = \text{cl}([0, 1])$.

b) Topological conjugacy

Definition 1.2 A continuous invertible function $h : M \to M'$, $y = h(x)$ with continuous inverse $x = h^{-1}(y)$ is called \underline{homeomorphism}.

Example 1.4 $h : [0, 1] \to [0, 1]$, $y = h(x) = \sin^2 \left(\frac{\pi}{2} x\right)$.

• $h(0) = 0$, $h(1) = 1$, and $h$ continuous implies $h$ surjective, i.e., $h([0, 1]) = [0, 1]$.

• $h'(x) = \frac{\pi}{2} \sin \left(\frac{\pi}{2} x\right) > 0$ if $0 < x < 1$. Thus $h$ is monotonic, i.e., $h$ is injective.

Thus $h$ is invertible. $h^{-1}(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$ is continuous and $h$ is a homeomorphism.

Definition 1.3 Two maps $f : M \to M$, $x_{n+1} = f(x_n)$ and $g : M' \to M'$, $y_{n+1} = g(y_n)$ are called \underline{topologically conjugate} if there exists a homeomorphism $h : M \to M'$, $y = h(x)$ such that $h \circ f = g \circ h$ (i.e. $h(f(x)) = g(h(x))$).

Example 1.5 The tent map $T(x) = 1 - |2x - 1|$ and the (Smale complete) logistic map $F_{r=4}(y) = 4y(1-y)$ are (topologically) conjugate via $h(x) = \sin^2 \left(\frac{\pi}{2} x\right)$ since $h : [0, 1] \to [0, 1]$ is a homeomorphism (see example 1.4) and

$$F_4(h(x)) = 4 \sin^2 \left(\frac{\pi}{2} x\right) \cos^2 \left(\frac{\pi}{2} x\right) = \sin^2(\pi x) = \sin^2 \left(\frac{\pi}{2} 2x\right) = \sin^2 \left(\frac{\pi}{2} (2 - 2x)\right) = h(T(x)).$$

Remark:

• If $(x_0, x_1, x_2, \ldots)$ denotes an orbit of $x_{n+1} = f(x_n)$ then $(y_0 = h(x_0), y_1 = h(x_1), y_2 = h(x_2), \ldots)$ yields an orbit of $g$ since $y_{n+1} = h(x_{n+1}) = h(f(x_n)) = g(h(x_n)) = g(y_n)$. In particular, $h$ maps periodic orbits of $f$ onto periodic orbits of $g$. 
• The relation \( f \sim g \) defined by “\( f \) is conjugate to \( g \)” defines an equivalence relation since
  (i) \( f \sim f \) (use \( h(x) = x \)), (ii) \( f \sim g \Rightarrow g \sim f \) (use \( h^{-1}(x) \)), and (iii) \( f_1 \sim f_2 \) and \( f_2 \sim f_3 \)
  \( \Rightarrow f_1 \sim f_3 \) (use \( h_2(h_1(x)) \)).

Example 1.6 The maps \( F_{r=4}(x) = 4x(1-x) \) and \( F_r(x) = rx(1-x) \), \( 0 < r < 1 \) are not
conjugate since \( F_r \) has no orbit of proper period two (example 1.2) and \( F_{r=4} \) has at least one
orbit of proper period two (example 0.5).

c) Topological transitivity

Consider the map \( f : [-1, 1] \to [-1, 1] \)
\[
y = f(x) = \frac{3\sqrt{3}}{2}x(1-x^2)
\]

![Graph of the function](image)

Obviously, positive (negative) initial conditions \( x_0 > 0 \) (\( x_0 < 0 \)) generate non-negative (non-
positive) orbits \( x_n \geq 0 \) (\( x_n \leq 0 \)) since \( f([0, 1]) = [0, 1] \) \( f([-1, 0]) = [-1, 0] \). We can “decompose” \( f \) into two maps \( f_1 : [0, 1] \to [0, 1] \) and \( f_2 : [-1, 0] \to [-1, 0] \).

Definition 1.4 A map \( f : M \to M \) is said to be topologically transitive if for any open subsets
\( A \subseteq M \) and \( B \subseteq M \) there exists some \( n > 0 \) such that \( f^n(A) \cap B \neq \emptyset \).

For any pair of points there exists a finite orbit which connects these points with arbitrary
precision.

Example 1.7 The map \( f(x) = \frac{3\sqrt{3}}{2}x(1-x^2) \) is not topologically transitive. Choose any
\( A \subseteq [-1, 0] \), say \( A = (-\frac{1}{2}, 0) \) and \( B \subseteq [0, 1] \), say \( B = (\frac{1}{2}, 1) \). Then \( f^n(A) \subseteq [-1, 0] \) as
\( f([-1, 0]) = [-1, 0] \) and \( f^n(A) \cap B = \emptyset \) for all \( n > 0 \).
**Proposition 1.2** Suppose two maps \( f : M \to M \) and \( g : M' \to M' \) are topologically conjugate, and \( f \) is topologically transitive (on \( M \)). Then \( g \) is topologically transitive (on \( M' \)).

**Proof:** Let \( A', B' \subseteq M' \) be open subsets (to show: \( g^n(A') \cap B' \neq \emptyset \) for some \( n > 0 \)).

\[
A = h^{-1}(A') \quad \text{and} \quad B = h^{-1}(B') \quad \text{are open subsets of} \quad M \quad (h^{-1} \text{ continuous!}). \quad f \text{ topological transitive implies: there exists some } n > 0 \text{ such that } f^n(A) \cap B \neq \emptyset. \text{ Thus } (as f \circ h^{-1} = h^{-1} \circ g \text{ implies } f^n \circ h^{-1} = h^{-1} \circ g^n): \emptyset \neq f^n(h^{-1}(A')) \cap h^{-1}(B') = h^{-1}(g^n(A')) \cap h^{-1}(B'). \text{ Therefore } (as h^{-1} \text{ invertible}) h^{-1}(g^n(A') \cap B') \neq \emptyset \text{ and } g^n(A') \cap B' \neq \emptyset.
\]

\[\square.\]

d) Stability and periodic points

Folklore: A fixed point \( x_* \) is said to be “stable” if \( x_n \to x_* \) (for “all” orbits); \( x_* \) is said to be “unstable” if \( x_n \not\to x_* \) (for “some” orbit).

**Definition 1.5** A fixed point \( x_* \) of a map \( f : M \to M \) is said to be attracting (“stable”) if there exists some \( \epsilon > 0 \) such that for all initial conditions \( x_0 \in U_\epsilon(x_*) \), \( x_n \to x_* \). \( x_* \) is said to be repelling (“unstable”) if there exists some \( \epsilon > 0 \) such that for any \( \delta < \epsilon \) there exists (at least one) initial condition \( x_0 \in U_\delta(x_*) \) such that \( x_k \not\in U_\epsilon(x_*) \) for some \( k > 0 \).

**Example 1.8** Consider \( f(x) = ax, f : \mathbb{R} \to \mathbb{R}, a \in \mathbb{R}, \, |a| \neq 1. \)