§0 Introduction

a) What is a dynamical system?

Example 0.1 Harmonic oscillator

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*}
\]

initial condition

\[
\begin{align*}
x(0) &= x_0, \\
v(0) &= v_0
\end{align*}
\]

solution

\[
\begin{align*}
x(t) &= x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \\
v(t) &= -\omega x_0 \sin(\omega t) + v_0 \cos(\omega t)
\end{align*}
\]

\[(x(t), v(t)) = \Phi_t(x_0, v_0)\]

phase space \(\mathbb{R}^2\)

flow \(\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2\) or \(\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2\).

(semi)group property

\[
\Phi_{t+s}(x_0, v_0) = \Phi_t(\Phi_s(x_0, v_0))
\]

Such a property is valid for fairly general differential equations

Example 0.2 Bouncing ball

\(t_n\): time of the \(n\)-th impact.

\(v_n\): velocity at the \(n\)-th impact

oscillating plate

\[w(t) = A\omega \cos(\omega t)\]
ballistic motion $n \mapsto n+1$: $h(t) = v_n t - \frac{1}{2} gt^2$

$$0 = v_n(t_{n+1} - t_n) - \frac{1}{2} g(t_{n+1} - t_n)^2 \Rightarrow t_{n+1} = t_n + \frac{2}{g} v_n$$

collision condition

$$v_{n+1} = v_n + 2w(t_n) = v_n + 2A\omega \cos\left(\omega t_n + \frac{2\omega}{g} v_n\right)$$

"elastic" "plate"

non-dimensional units: $\Theta_n = \omega t_n$, $u_n = \frac{2\omega}{g} v_n$, $\beta = 2A\omega^2$

$$\Theta_{n+1} = \Theta_n + u_n$$

"standard map"

$$u_{n+1} = u_n + \beta \cos(\Theta_n + u_n) \quad (\Theta_{n+1}, u_{n+1}) = F(\Theta_n, u_n)$$

$$(\Theta_1, u_1) = F(\Theta_0, u_0)$$

$$(\Theta_2, u_2) = F(\Theta_1, u_1) = F(F(\Theta_0, u_0)) = F^{(2)}(\Theta_0, u_0) \quad 2\text{nd iterate}$$

$$(\Theta_n, u_n) = F^{(n)}(\Theta_0, u_0) = \Phi_n(\Theta_0, u_0) \quad \text{"flow"}$$

**Definition 0.1** A dynamical system consists of a phase space (state space) $M$ (here normally $M \subseteq \mathbb{R}^n$) and a family of transformations $\Phi_t : M \to M$ where the “time” $t$ may be either discrete ($t \in \mathbb{N}$) or continuous ($t \in \mathbb{R}$). The family obeys

i) $\Phi_0(x) = x$ (for all $x \in M$), and

ii) $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ (for all $s, t$ and $x \in M$).

**Definition 0.2** The sequence $(\Phi_n(x_0), n = 0,1,2,\ldots)$ or the curve $\{\Phi_t(x_0), t \in \mathbb{R}\}$ is called orbit (trajectory) with initial condition $x_0$.

**Remark:** Poincaré maps.

Suppose $\Phi_t$ denotes the flow of a differential equation $\dot{x} = f(x)$ where $x \in \mathbb{R}$. Consider an $n-1$ dimensional “surface” such that an orbit intersects the surface successively (a so called “cross section”).

The sequence of subsequent crossings $y_n \mapsto y_{n+1}$ defines the Poincaré map $P : \Sigma \to \Sigma$ of the flow.
Example 0.3 Harmonic oscillator

\[ \Sigma = \{(x,v) : v = 0, x \geq 0\} \]

\[ (x_n, v_n = 0) \mapsto (x_{n+1}, v_{n+1} = 0) \]

\[ x(t) = x_n \cos(\omega t) + 0 \]

\[ v(t) = -\omega x_n \sin(\omega t) + 0 \]

Crossing condition: \( v(t) = 0, x(t) \geq 0 \Rightarrow t = \frac{2\pi}{\omega} \).

\[ \Rightarrow x_{n+1} = x \left( \frac{2\pi}{\omega} \right) = x_n \Rightarrow x_{n+1} = P(x_n) = x_n \quad \text{("trivial" identity map)} \]

b) Limit sets

The flow \( \Phi \) is normally difficult to analyse. Focus on simpler properties which are still relevant for the long time behaviour. For instance:

\[ x_{n+1} = r x_n (1 - x_n) \]

\[ r = 3.83 \]

Definition 0.3 A point \( x_\omega \in M \) is called \( \omega \)-limit point of an orbit, if for any (large time) \( T \geq 0 \) and (small distance) \( \varepsilon > 0 \) there exists (at least one time) \( t \geq T \) such that \( |\Phi_t(x_0) - x_\omega| < \varepsilon \) (short: \( \forall T \geq 0, \exists \varepsilon > 0, \exists t \geq T, |\Phi_t(x_0) - x_\omega| < \varepsilon \)). The set of all \( \omega \)-limit points is called \( \omega \)-limit set of the orbit.

Example 0.4 Suppose \( \Phi_n(x_0) = (-1)^n + \frac{1}{n} \)

Then \( S = \{-1, 1\} \) since

i) \( x_\omega = 1 \): Choose \( n = 2k \) (even). Then

\[ |\Phi_{2k}(x_0) - 1| = \frac{1}{2k} < \varepsilon \quad \text{if} \quad k > \frac{1}{2\varepsilon} \]