§4 Symbolic dynamics

We want to study dynamical behaviour which is beyond fixed points and periodic solutions, i.e., more complex long time behaviour (so called limit sets). To address the fundamental issues related with this question we resort to one-dimensional and two-dimensional maps where there is a chance to develop the basis of a theory with reasonable effort, and to clarify the mechanisms which are relevant in more general dynamical systems. Our first goal will be to “count” orbits.

Definition 4.1 Let \( f : M \to M, x_{n+1} = f(x_n) \) denote a discrete dynamical system on a phase space \( M \), and let \( f^{(n)} = f \circ f \circ \ldots \circ f \) denote the \( n \)-th iterate (i.e., \( f^{(n)}(x) = f(f(\ldots f(x))) \)).

i) The sequence \((x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots)\) is called orbit with initial condition \( x_0 \).

ii) \( x_* \in M \) is called fixed point if \( x_* = f(x_*) \) (i.e. a constant orbit \((x_*, x_*, x_*, \ldots)\)).

ii) The (finite) sequence \((x_0, x_1, \ldots, x_{p-1})\) is called orbit of period \( p \) if

\[
  x_1 = f(x_0), \quad x_2 = f(x_1), \quad x_{p-1} = f(x_{p-2}), \quad x_0 = f(x_{p-1})
\]

that means \( x_{\ell} = f^{(p)}(x_{\ell}), \ell = 0, 1, \ldots, p-1 \) Each point of the sequence is called periodic point of period \( p \). \( p \) is called proper period (prime period) if all elements of the sequence are distinct.

a) Bernoulli shift map

Consider the map \( B : [0, 1) \to [0, 1) \) defined by

\[
  x_{n+1} = B(x_n) = 2x_n \mod 1 = \begin{cases} 
    2x_n & \text{if } 0 \leq x_n < 1/2 \\
    2x_n - 1 & \text{if } 1/2 \leq x_n < 1 
  \end{cases}
\]

The map is continuous (and the domain closed) if \([0, 1)\) is identified with the unit circle

\[
  S^1 = \{ z \in \mathbb{C} : |z| = 1 \} = \{ z = e^{2\pi i x} : x \in [0, 1) \}
\]
since \( b(z) = z^2 \) maps \( S^1 \) to \( S^1 \) and

\[
b(z) = z^2 = e^{2\pi i 2x} = e^{2\pi i (2x-1)} = e^{2\pi i B(x)}.
\]

Distance (metric) on \( S^1 \):

\[
|z - z'| = |e^{2\pi i x} - e^{2\pi i x'}| = |e^{\pi i (x-x')} - e^{-\pi i (x-x')}| = 2|\sin(\pi (x - x'))|
\]

(not \(|x - x'|!\)).

- Fixed points: \( x_* = B(x_*) = 2x_* \mod 1 = 2x_* - k, k \in \mathbb{Z} \). Thus \( x_* = k \) (\( \in \mathbb{Z} \)) and \( x_* = 0 \) (since \( x_* \in [0, 1) \)).

- Period-two points: \( x_* = B^{(2)}(x_*) = 2(2x_* \mod 1) \mod 1 = 2(2x_* - k_1) - k_2 = 4x_* - k, (k_{1/2} \in \mathbb{Z}, k \in \mathbb{Z}) \). Thus \( x_* = \frac{k}{3} \), i.e., \( x_* = 0 \) (fixed point), or \( x_* = \frac{1}{3} \), or \( x_* = \frac{2}{3} \) \((x_0 = \frac{1}{3}, x_1 = \frac{2}{3})\) is a proper period-two orbit.

- Period-\( n \) points: \( x_* = B^{(n)}(x_*) = 2^n x_* \mod 1 = 2^n x_* - k \). Thus \( x_* = \frac{k}{2^n - 1}, k = 0, 1, 2, \ldots, 2^n - 2 \) and the set of period-\( n \) points reads

\[
\text{Per}_n(B) = \{x \in [0, 1) : B^{(n)}(x) = x\} = \left\{\frac{k}{2^n - 1} : 0 \leq k \leq 2^n - 2\right\}
\]

Consider the Bernoulli shift map \( x_{n+1} = 2x_n \mod 1 \) (on \([0, 1)\)) and write numbers \( x \in [0, 1) \) as binary fractions, i.e.,

\[
x = \frac{1}{2} \sigma_0 + \frac{1}{4} \sigma_1 + \frac{1}{8} \sigma_2 + \ldots \quad (\cong 0.\sigma_0\sigma_1\sigma_2\ldots) \quad \sigma_k \in \{0, 1\}
\]

e.g. \( \frac{1}{3} = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{8} + 1 \cdot \frac{1}{16} + \ldots \), i.e., 0.010101 . . . How does \( B(x) \) “look like” if binary fractions are used?

\[
2x = \sigma_0 + \frac{1}{2} \sigma_1 + \frac{1}{4} \sigma_2 + \ldots \quad (\cong \sigma_0.\sigma_1\sigma_2\ldots)
\]

\[
B(x) = 2x \mod 1 = \frac{1}{2} \sigma_1 + \frac{1}{4} \sigma_2 + \frac{1}{8} \sigma_3 + \ldots \quad (\cong 0.\sigma_1\sigma_2\sigma_3\ldots)
\]

Thus

\[
\begin{align*}
x & \xrightarrow{B} x' = B(x) = 2x \mod 1 \\
\downarrow & \downarrow \\
0.\sigma_0\sigma_1\sigma_2\ldots & \xrightarrow{S} 0.\sigma_1\sigma_2\sigma_3\ldots
\end{align*}
\]

"almost a bijection"

"symbol shift"

Properties of the symbolic dynamics:
• every periodic symbol sequence yields a periodic orbit, e.g. $p = 2$

$$0.010101\ldots \xmapsto{S} 0.101010\ldots \xmapsto{S} 0.010101\ldots$$

i.e. a fixed point of $S^{(2)}$.  

• For any two points $x$ and $x'$ let the binary fractions be given by $0.\sigma_0\sigma_1\sigma_2\ldots$ and $0.\sigma'_0\sigma'_1\sigma'_2\ldots$. The point $y$ with binary fraction $0.\sigma_0\sigma_1\ldots\sigma_{N-1}\sigma'_0\sigma'_1\sigma'_2\ldots$ differs from $x$ by at most

$$|x - y| = \left| \frac{1}{2^{N+1}}(\sigma_N - \sigma'_0) + \frac{1}{2^{N+2}}(\sigma_{N+1} - \sigma'_1) + \ldots \right|$$

$$\leq \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \ldots = \frac{1}{2^N}$$

which can be made as small as we wish, say $|x - y| < \varepsilon$, if $N$ is large. But after $N$ iterations, i.e., after $N$ symbol shifts, $y$ is mapped to $x'$. Thus we have an orbit which starts near $x$ and ends at $x'$ (that means the Bernoulli map is topological transitive).

• Tiny glitch: the map $0.\sigma_0\sigma_1\sigma_2\ldots \mapsto x$ is not bijective since, e.g., both $0.01111\ldots$ and $0.10000\ldots$ are mapped to $\frac{1}{2}$. But that does not really affect the previous considerations (there are only few such points, e.g., the “discontinuity”).

• How to generalise the considerations? $\sigma_0$ distinguishes between $x_0 < \frac{1}{2}$ and $x_0 \geq \frac{1}{2}$. Take the partition $I_0 = [0, \frac{1}{2})$, $I_1 = [\frac{1}{2}, 1)$ of $[0, 1)$. If $0.\sigma_0\sigma_1\sigma_2\ldots$ denotes the symbol sequence of $x_0$ then $x_0 \in I_{\sigma_0}$, $x_1 \in I_{\sigma_1}$ and so on. The recipe $B^{(k)}(x) \in I_{\sigma_k}$ generates the symbol sequence of $x$!

b) Expansive Markov maps

**Definition 4.2** Let $I = [a,b]$ be a closed interval. $|I| = |b - a|$ denotes the size of the interval, $\text{int}(I) = (a,b)$ the interior of $I$. A collection of closed intervals $\{I_0, I_1, \ldots, I_{N-1}\}$ is called partition of $I$ if

i) $I = \bigcup_{k=0}^{N-1} I_k$, and

\[ \text{int}(I_k) \cap \text{int}(I_\ell) = \emptyset \text{ if } k \neq \ell. \]
Definition 4.3 A map \( f : I \rightarrow I \) is called Markov map if there exists a partition \( \{I_0, I_1, \ldots, I_{N-1}\} \), a so called Markov partition, such that for all \( k, \ell = 0, \ldots, N-1 \)

i) either \( f(\text{int}(I_k)) \cap \text{int}(I_\ell) = \emptyset \)

ii) or \( \text{int}(I_\ell) \subseteq f(\text{int}(I_k)) \).

Example 4.1 \( f : [0, 1] \rightarrow [0, 1] \):

\[
N = 2 \\
I_0 = [0, \frac{1}{2}], \text{int}(I_0) = (0, \frac{1}{2}) \\
I_1 = [\frac{1}{2}, 1], \text{int}(I_1) = (\frac{1}{2}, 1) \\
\{I_0, I_1\} \text{ is a partition.}
\]

- \( f(\text{int}(I_0)) = (0, 1) \), that means, \( f(\text{int}(I_0)) \supseteq \text{int}(I_0) \) and \( f(\text{int}(I_0)) \supseteq \text{int}(I_1) \).

- \( f(\text{int}(I_1)) = (0, \frac{1}{2}) \) that means \( f(\text{int}(I_1)) \supseteq \text{int}(I_0) \) and \( f(\text{int}(I_1)) \cap \text{int}(I_0) = \emptyset \).

No “transition” \( I_1 \rightarrow I_1 \) is possible.

Example 4.2 \( f : [0, 1] \rightarrow [0, 1] \)

\[
I_0 = [0, \frac{1}{2}], I_1 = [\frac{1}{2}, 1] \text{ is not a Markov partition since } f(\text{int}(I_1)) = (0, \frac{3}{4}), \text{i.e., } f(\text{int}(I_1)) \cap \text{int}(I_1) = (\frac{1}{2}, \frac{3}{4}) \neq \emptyset, \text{ but int}(I_1) \text{ is not contained in } f(\text{int}(I_1)).
\]

Is there another partition which is a Markov partition?

Markov maps map boundary points of the partition on boundary points.
**Definition 4.4** The \( N \times N \) matrix defined by

\[
A_{k\ell} = \begin{cases} 
1 & \text{if } f(\text{int}(I_k)) \supseteq \text{int}(I_\ell) \\
0 & \text{if } f(\text{int}(I_k)) \cap \text{int}(I_\ell) = \emptyset
\end{cases}
\]

is called the topological transition matrix of the Markov map \( f \) ("\( A_{k\ell} = 1 \) if the transition \( I_k \to I_\ell \) is permitted")

**Example 4.3** (see example 4.1)

\[
\begin{align*}
I_0 &\to I_0 \\
I_0 &\to I_1 \\
I_1 &\to I_0 \\
I_1 &\not\to I_1
\end{align*}
\]

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]

**Definition 4.5** A Markov map is said to be an expansive (expanding) Markov map if \( f \) is smooth on each \( \text{int}(I_k) \) and \( |f'(x)| \geq \lambda > 1 \) for \( x \in \text{int}(I_k) \), \( k = 0, \ldots, N-1 \).

**Remark:** It is sufficient to require expansiveness for some iterate, say \( |(f^{(n)})'(x)| \geq \lambda > 1 \).

**Example 4.4** (see example 4.3)

\[
|f'(x)| = 2 \text{ if } x \in \text{int}(I_0) \text{ but } |f'(x)| = 1 \text{ if } x \in \text{int}(I_1). \text{ Nevertheless, } |(f^{(2)})'(x)| = \quad \quad \quad \quad \quad \quad \quad \quad \\
|f'(f(x))||f'(x)| = 2 \cdot 1 \text{ if } x \in \text{int}(I_1), \text{ as } \quad \quad \quad \quad \quad \quad \quad \quad \\
x \in I_1 \text{ implies } f(x) \in I_0, \text{ i.e., } f'(x) = 1 \text{ and } \quad \quad \quad \quad \quad \quad \quad \quad \\
f'(f(x)) = -2. \text{ In addition } |(f^{(2)})'(x)| \geq 2 \text{ if } \quad \quad \quad \quad \quad \quad \quad \quad \\
x \in \text{int}(I_0). \text{ Thus the map is still expansive.}
\]

**Main idea:** Consider a Markov map \( f \) with Markov partition \( \{I_0, I_1, \ldots, I_{N-1}\} \) and symbols \( \sigma \in \{0, 1, \ldots, N-1\} \). Assign to \( x_0 \in I \) the symbol sequence \( .\sigma_0\sigma_1\sigma_2\ldots \) according to the rule \( x_k = f^{(k)}(x_0) \in I_{\sigma_k} \). Then \( x_n \) has symbol sequence \( .\sigma_n\sigma_{n+1}\ldots \).
Definition 4.6 Let $f$ be a Markov map with transition matrix $A$. A symbol sequence $\sigma_0\sigma_1\sigma_2\ldots$ is called admissible if $A_{\sigma_k\sigma_{k+1}} = 1$, $k \geq 0$.

Example 4.5 Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

- $0.110010\ldots = .\sigma_0\sigma_1\sigma_2\ldots$ is not admissible since $A_{\sigma_1\sigma_2} = A_{11} = 0$.
- $0.100010\ldots$ is admissible.

All symbol sequences which contain a pair “11” are not admissible since $A_{11} = 0$.

Proposition 4.1 Let $f : I \to I$ be a continuous expansive Markov map with transition matrix $A$. For any admissible symbol sequence $\sigma_0\sigma_1\sigma_2\ldots$ there exists a unique point $x \in I$ such that $f^{(k)}(x) \in I_{\sigma_k}$ (for any $k \geq 0$). The mapping from admissible symbol sequence to the domain $I$ is onto.

Proof: Main idea: for the tent map $T : [0,1] \to [0,1]$

$$T(x) = 1 - |1 - 2x|$$

$I_0 = [0, \frac{1}{2}]$ and $I_1 = [\frac{1}{2}, 1]$ defines a Markov partition with transition matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

i.e., any symbol sequence is admissible.

Fix a symbol sequence, say $.101\ldots$.
Construct a sequence of nested closed intervals

\[ I_1 \supseteq I_{10} \supseteq I_{101} \supseteq \ldots \]

"length" \( \frac{1}{2} \) \( \frac{1}{4} \) \( \frac{1}{8} \) \( \ldots \) \Rightarrow \text{point } x \text{ such that } T^{(k)}(x) \in I_{\sigma_k} \)

Since \( \{I_0, I_1\} \) is a partition, for any \( z \in [0, 1] \) there exists a symbol sequence such that \( T^{(k)}(z) \in I_{\sigma_k} \). Then (if \( x \) denotes the point according to the previous construction) \( |T^{(n)}(z) - T^{(n)}(x)| = 2^n |z - x| \) for any \( n \), i.e. \( z = x \). Thus, the symbolic dynamics is onto.

Structure of the proof: let \( .\sigma_0\sigma_1\sigma_2 \ldots \) be an admissible symbol sequence.

- admissible implies \( \text{int}(I_{\sigma_0}) \subseteq f(\text{int}(I_{\sigma_0})) \).
- continuity of \( f \) implies \( I_{\sigma_1} \subseteq f(I_{\sigma_0}) \). Thus \( I_{\sigma_0\sigma_1} = I_{\sigma_0} \cap f^{-1}(I_{\sigma_1}) \neq \emptyset \), \( I_{\sigma_0\sigma_1} \) is a closed interval, and \( f : I_{\sigma_0\sigma_1} \to I_{\sigma_1} \) is one to one! Of course \( I_{\sigma_0\sigma_1} \subseteq I_{\sigma_0} \).
- \( f \) expansive means \( |I_{\sigma_1}| = |f(I_{\sigma_0\sigma_1})| \geq \lambda |I_{\sigma_0\sigma_1}| \). Thus \( |I_{\sigma_0\sigma_1}| \leq \frac{1}{\lambda} |I_{\sigma_1}| (\leq \frac{1}{\lambda^2} |I|) \).
- by induction (e.g. \( \{I_{\sigma_0\sigma_1}\} \) is again a Markov partition): \( I_{\sigma_0} \supseteq I_{\sigma_0\sigma_1} \supseteq I_{\sigma_0\sigma_1\sigma_2} \supseteq \ldots \) is a sequence of nested closed intervals (the so called "cylinder sets") where \( |I_{\sigma_0}| \leq \frac{1}{\lambda} |I|, |I_{\sigma_0\sigma_1}| \leq \frac{1}{\lambda^2} |I|, \ldots \). Thus \( \cap_{n \geq 0} I_{\sigma_0\sigma_1\ldots\sigma_n} \) contains one point \( x \) and \( f^{(k)}(x) \in I_{\sigma_k} \) for \( k \geq 0 \).
- uniqueness follows from expansivity: if both \( x \) and \( x' \) obey \( f^{(k)}(x) \in I_{\sigma_k} \) and \( f^{(k)}(x') \in I_{\sigma_k} \) for \( k \geq 0 \), then \( |I| \geq |f^{(k)}(x) - f^{(k)}(x')| \geq \lambda^k |x - x'| \Rightarrow x - x' = 0 \) as \( \lambda^k \to \infty \).
onto follows from expansivity and the sets being a partition.

\[ \square. \]

**Example 4.6** (see example 4.1)

Expansive Markov map with transition matrix (see example 4.3)

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

Periodic orbits (admissible symbol sequences)

- \( .0000 \ldots \) fixed point
- \( .010101 \ldots .1010 \ldots \) proper period two
- \( .001001 \ldots .010010 \ldots .100100 \ldots \) proper period three

Computation of the period-two orbit: \( .010101 \ldots \triangleq x_0 \in I_0, .101010 \ldots \triangleq x_1 \in I_1 \)

\[
\begin{align*}
x_1 &= f(x_0) = 1 - 2x_0 \\
x_0 &= f(x_1) = x_1 - \frac{1}{2}
\end{align*}
\]

\[ x_1 = \frac{2}{3}, \quad x_0 = \frac{1}{6} \]

**Remark:** The boundaries of the Markov partition are mapped onto boundary points, i.e., the boundary is contained in the stable set of a periodic orbit. That is a feature which to some extend prevails in higher dimensional diffeomorphism, like the cat map. Segments of the stable and unstable manifold determine a Markov partition and the symbolic dynamics now results in a two-sided symbol shift. In fact corresponding statements are true for general hyperbolic dynamical systems.
Summary: symbolic dynamics

\[ f : \quad x \quad \mapsto \quad f(x) \]

\[ \quad \uparrow \quad \quad \uparrow \quad \text{semi conjugacy} \]

\[ S : \ .\sigma_0\sigma_1\sigma_2\ldots \quad \mapsto \quad .\sigma_1\sigma_2\sigma_3\ldots \]