§3 Bifurcations

Hyperbolic fixed points, and more generally hyperbolic systems are structurally stable, i.e., the system does not change its dynamical behaviour if small perturbations are applied. We are now going to investigate the case where a dynamical system undergoes a transition. The simplest and already illuminating case related with such non-hyperbolic behaviour will be the change of stability of fixed points. In fact, the study yields a classification of instabilities and related universal behaviour.

a) Centre manifolds

If $x_*$ denotes a hyperbolic fixed point the dynamics in a neighbourhood can be captured by the linear part of the dynamics, i.e., by stable and unstable manifolds which are tangential to the linear subspaces of the Jacobian matrix. In fact, using linear superposition the entire flow can be decomposed into parts associated with the stable and unstable subspaces (cf. e.g. section 1b). A similar statement applies if the Jacobian has eigenvalues with vanishing real part. In addition to stable and unstable manifolds a so called centre manifold appears which contains the dynamics related with vanishing eigenvalues (to be more precise: vanishing real part).

**Proposition 3.1 ("Centre manifold Theorem")** Suppose the differential equation $\dot{x} = f(x)$ induces a smooth flow with fixed point $x_*$. Let $E^s$, $E^u$, and $E^c$ denote the linear eigenspaces of the Jacobian matrix $Df(x_*)$ which correspond to eigenvalues $\text{Re}(\lambda) < 0$, $\text{Re}(\lambda) > 0$, and $\text{Re}(\lambda) = 0$ respectively. Then there exist smooth invariant manifolds $W^s$ and $W^u$ tangent to $E^s$ and $E^u$ at $x_*$ and an invariant manifold $W^c$ tangent to $E^c$. The stable and unstable manifolds $W^s$ and $W^u$ are unique.

Roughly speaking, in the non-hyperbolic case it is sufficient to study the dynamics on the centre manifold only (as the dynamics on the other manifolds is exponential in time). In particular, if the unstable manifold is empty (i.e. $\text{Re}(\lambda) \leq 0$) the dynamics relaxes exponentially to the centre manifold and the entire motion can be captured by a lower-dimensional dynamical system ("adiabatic elimination").

**Example 3.1** Nonlinear oscillator
Equations of motion
\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -v - x^3
\end{align*}
\]
Fixed point \((x_*, v_*) = (0, 0)\).

Variational equation
\[
\begin{pmatrix}
\delta \dot{x} \\
\delta \dot{v}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\delta x \\
\delta v
\end{pmatrix}
\]
Linear eigenspaces
\[
\lambda^s = -1, \quad \nu^s = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \lambda^c = 0, \quad \nu^c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
Centre manifold
\[
v = h(x) = 0 \cdot x + a_2 x^2 + a_3 x^3 + \ldots
\]
Invariance
\[
\dot{v} = -h(x) - x^3 = h'(x) \dot{x} = h'(x) h(x)
\]
Thus \(a_2 = 0, a_3 = -1\) and
\[
\dot{x} = v = h(x) = -x^3 + \ldots
\]
In particular, the solution \(x(t)\) tends to zero as \(t \to \infty\) ("nonlinear stability").

b) Codimension-one bifurcations of fixed points

Let \(\dot{x} = f_\alpha(x)\) denote a differential equation in \(\mathbb{R}^N\) where the right hand side depends on a parameter \(\alpha \in \mathbb{R}\). Suppose that for parameter value \(\alpha = \alpha_0\) the equation has a (non-hyperbolic) fixed point \(x_*\) with a single eigenvalue zero of the Jacobian matrix \(Df_{\alpha_0}(x_*)\), i.e.,
\[
f_{\alpha_0}(x_*) = 0 \quad \text{and} \quad \det Df_{\alpha_0}(x_*) = 0.
\]
These \(N + 1\) equations give one algebraic condition for the parameter (after the \(N\) coordinates of the fixed point have been eliminated), i.e., it determines a codimension one manifold in parameter space (e.g. a point in \(\mathbb{R}\)).
If we suppose that there is just a single eigenvalue with vanishing real part, the dynamics in a neighbourhood of the fixed point can be captured by an equation on a one-dimensional centre manifold. If we chose the coordinates such that the origin is the fixed point the equation of motion on the centre manifold reads, \( \dot{y} = -y^2 + \ldots \), as the linear part vanishes (and any coefficient of the quadratic contribution can be absorbed in rescaling the time, as long as the coefficient does not vanish). Dots indicate higher order terms of a Taylor series expansion (actually an asymptotic series).

If we consider parameter values close to \( \alpha_0 \) then the equation of motion on the centre manifold will contain additional small contributions, \( \dot{y} = a_0 + a_1 y - y^2 + \ldots \), a so called unfolding. Using a transformation \( z = y + c \) one obtains \( \dot{z} = a_0 + a_1 c - c^2 + (a_1 - 2c)z - z^2 + \ldots \), i.e., by the appropriate choice \( c = a_1/2 \) the linear term can be eliminated and one ends up with the so called normal form of the bifurcation

\[
\dot{z} = \mu - z^2.
\]

Here the parameter \( \mu \) is a function of the original parameter \( \alpha \) of the system. The expression tells us that there is no fixed point for \( \mu < 0 \) and a pair of stable and unstable fixed points \( x_\ast = \pm \sqrt{\mu} \) for \( \mu > 0 \). That means by crossing the bifurcation point \( \mu = 0 \) in parameter space a pair of fixed points is generated/destroyed ("saddle-node bifurcation"). The reasoning can be made rigorous with a few additional technical assumptions (mainly to guarantee nonzero coefficients in the normal form).

**Proposition 3.2 ("Saddle-node bifurcation")** Let \( \dot{x} = f_\alpha(x) \) denote a differential equation depending on a parameter \( \alpha \). When \( \alpha = \alpha_0 \) assume that there exists a fixed point \( x_\ast \) such that

1. \( Df_{\alpha_0}(x_\ast) \) has a simple eigenvalue 0 with right eigenvector \( v \) and left eigenvector \( w \). All other eigenvalues have nonzero real part.
2. \( w \cdot \frac{\partial f_{\alpha_0}(x_\ast)}{\partial \alpha_0} \neq 0 \).
(SN3): \( wD^2 f_{\alpha_0}(x_*)(v, v) \neq 0 \).

Then there exists a smooth curve of fixed points \( \xi_*(\alpha) \) with \( \xi_*(\alpha_0) = x_* \). Depending on the signs of the expressions in (SN2) and (SN3) there are no fixed points near \( x_* \) when \( \alpha < \alpha_0 \) (or \( \alpha > \alpha_0 \)). The two fixed points near \( \alpha_0 \) are hyperbolic.

**Example 3.2** Forced van der Pol oscillator: the condition for the fixed point can be written as (see example 1.4)

\[
\begin{pmatrix}
0 \\
\gamma
\end{pmatrix} =
\begin{pmatrix}
1 - u_*^2 - v_*^2 & -\sigma \\
\sigma & 1 - u_*^2 - v_*^2
\end{pmatrix}
\begin{pmatrix}
u_* \\
v_*
\end{pmatrix}.
\]

If one considers this expression as a linear system and solves for \( u_* \) and \( v_* \) we obtain for \( r_* = \sqrt{u_*^2 + v_*^2} \) the condition

\[
r_*^2 = \frac{\gamma^2}{(1 - r_*^2)^2 + \sigma^2}.
\]

The Jacobian matrix is given by

\[
Df(x_*) =
\begin{pmatrix}
1 - 3u_*^2 - v_*^2 & -\sigma - 2u_*v_* \\
\sigma - 2u_*v_* & 1 - 3v_*^2 - u_*^2
\end{pmatrix}.
\]

The condition for a vanishing eigenvalue thus results in

\[
0 = \det(Df(x_*)) = (1 - r_*^2)^2 - 2r_*^2(1 - r_*^2) + \sigma^2.
\]

Both conditions are now easily rearranged to result in

\[
\begin{align*}
\gamma^2 &= 2r_*^4(1 - r_*^2) \\
\sigma^2 &= (3r_*^2 - 1)(1 - r_*^2)
\end{align*}
\]

These expressions can be viewed as a parametric representation of a curve in the \( \gamma-\sigma \) parameter plane with the curve parameter \( r_*^2 \) obeying the constraint \( 1/3 \leq r_*^2 \leq 1 \).

Using the calculation for \( \sigma = 0 \) (see example 2.2) we are as well able to identify the region in parameter space where the fixed point pair is created, without any further computation.
The second possibility by which a hyperbolic fixed point can change stability is via a complex conjugate pair of eigenvalues (i.e. vanishing real part but non-vanishing imaginary part). Suppose $\alpha_0$ is a parameter value where such an instability takes place. The (non-hyperbolic) fixed point and the parameter obeys

$$f_{\alpha_0}(x_*) = 0 \quad \text{and} \quad \det(Df_{\alpha_0}(x_*) - i\omega) = 0$$

where $\omega \neq 0$ denotes the imaginary part of one of the eigenvalues. These $N+2$ real equations for the unknowns $(x_*, \omega, \mu_0)$ give again one algebraic constraint for the parameter value, i.e. the equations determine a codimension one manifold in parameter space where the instability happens.

A complex conjugate pair implies a two-dimensional centre manifold (cf. as well example 1.6). The corresponding equation of motion on the centre manifold and the unfolding in a neighbourhood of the bifurcation point requires some substantial computations (but is rather straightforward). The resulting two-dimensional differential equation is most conveniently written down in polar coordinates $(x = r \cos(\varphi), y = r \sin(\varphi))$

$$\dot{r} = \mu r - \sigma r^3 + \ldots, \quad \dot{\varphi} = \omega + \ldots$$

where $\ldots$ indicates higher order terms in the radial variable $r$. $\mu$ denotes the unfolding parameter and the value of $\sigma$ is either $+1$ or $-1$ (i.e. there are two different types of the bifurcation).

Let us first consider the case $\sigma = 1$ (the so called supercritical case). The fixed point $r = 0$ (i.e. $x = y = 0$) is stable for $\mu < 0$ and unstable for $\mu > 0$. If $\mu > 0$ then there is a (stable) periodic solution $r = \sqrt{\mu}$, $\varphi = \omega t$, i.e., $x(t) = \sqrt{\mu} \cos(\omega t)$, $y(t) = \sqrt{\mu} \sin(\omega t)$, a so called limit cycle.

In the other case $\sigma = -1$ (the so called subcritical case) the stability properties of the fixed points are the same, but the system develops an unstable periodic solution $r = \sqrt{-\mu}$, $\varphi = \omega t$, i.e., $x(t) = \sqrt{-\mu} \cos(\omega t)$, $y(t) = \sqrt{-\mu} \sin(\omega t)$ for $\mu < 0$.

The instability with this characteristics, i.e., where the stability change of the fixed point
gives rise to a limit cycle, is called a Hopf bifurcation. The result can be made rigorous with a few additional technical assumptions.

**Example 3.3** Forced van der Pol oscillator:

The condition for a Hopf bifurcation in a two-dimensional system, i.e., a complex conjugate imaginary pair of eigenvalues results in \( \text{Tr}(Df(x_*)) = 0 \) and \( \det(Df(x_*)) > 0 \).

With the Jacobian matrix (see example 3.2) we thus obtain

\[
0 = 2 - 4r_*^2, \quad (1 - r_*^2)^2 - 2r_*^2(1 - r_*^2) + \sigma^2 > 0.
\]

With the condition for the fixed point (see example 3.2) we end up with

\[
r_*^2 = \frac{1}{2}, \quad \sigma^2 > \frac{1}{4}, \quad \gamma^2 = \frac{1}{2} \left( \frac{1}{4} + \sigma^2 \right)
\]

**Remark:** We can finally supplement parts of this bifurcation diagram by taking the considerations from example 2.2 into account. The example tells us (essentially) that within the domain bounded by the saddle node bifurcation line the flow contains a stable node/focus (s.n.) an unstable node/focus (u.n.), and a saddle (sd.). Furthermore, example 1.4 tells us that the flow possesses a stable limit cycle (s.l.c.) at \( \gamma = 0 \). That suggests that the limit cycle is destroyed at the Hopf bifurcation line in a subcritical Hopf bifurcation leaving a stable focus (s.f.) behind. We have been therefore able to gain substantial insight into the phase portrait (up to topological transformations) without actually integrating the underlying differential equations. There are still particular points in the bifurcation diagram, e.g., the point where the Hopf bifurcation line terminates at the saddle node bifurcation, or the cusp formed by the merging of two saddle node bifurcation lines. These degenerate cases are related with higher codimension bifurcations.

**c) Global bifurcations**

So far we have considered how a local change in the shape (in the topology) of a phase portrait (of a flow) is related with the change of stability of fixed points. There are
other mechanisms which can trigger global changes affecting not just a small neighbourhood, when a parameter is changed. We just illustrate the case of a so called homoclinic bifurcation for two-dimensional flows.

Consider a two-dimensional differential equation with saddle point, i.e., a hyperbolic fixed point with a stable and an unstable manifold. An orbit is called homoclinic if stable and unstable manifold coincide.

**Example 3.4**: “The fish”: Consider the following nonlinear oscillator

\[
\dot{x} = v, \quad \dot{v} = 2x - 3x^2
\]

There are two fixed points \((x^a_*, v^a_*) = (0, 0)\) and \((x^b_*, v^b_*) = (2/3, 0)\). The Jacobian matrix reads

\[
Df(x_*) = \begin{pmatrix}
0 & 1 \\
2 - 6x_* & 0
\end{pmatrix}
\]

i.e., \((x^a_*, v^a_*)\) is a saddle (eigenvalues \(\pm \sqrt{2}\) with stable and unstable eigenvectors \((1, \mp \sqrt{2})\)).

It is fairly easy to show that “the energy”

\[
H = v^2/2 - x^2 + x^3
\]

is a conserved quantity (for instance \(\dot{H} = vv + (-2x + 3x^2)x = v(2x - 3x^2) + (-2 + 3x^2)v = 0\)). Thus, the condition

\[
H = v^2 - x^2 + x^3 = \text{const.}
\]

determines the orbits in phase space (i.e., the orbits are the level curves of the function \(H\)). The curve

\[
0 = v^2 - x^2 + x^3
\]

gives a homoclinic orbit.

The example is atypical in the sense that many orbits are closed curves (the system is a two-dimensional Hamiltonian dynamical system).

Consider a two-dimensional flow which depends on a parameter \(\mu\). Suppose that at some value \(\mu = \mu_0\) the flow contains a homoclinic orbit and that when changing the parameter the stable and unstable manifolds miss each other (with different orientation for \(\mu < \mu_0\) and \(\mu > \mu_0\), respectively).
At $\mu_0$ the flow changes its topological structure (i.e. this is an example of a global bifurcation). But even more, for $\mu > \mu_0$ the form of the phase portrait requires the existence of a periodic orbit, i.e., the global bifurcation requires the creation of non-constant long time behaviour. Homoclinic bifurcations become even more subtle in higher dimensional phase spaces or for two dimensional maps (cf. the cat map), as this constitutes the main mechanism for the creation of chaos.

**Example 3.5** : “The dissipative fish”: consider the previous nonlinear oscillator with an additional “suitable damping force”

$$\dot{x} = v, \quad \dot{v} = 2x - 3x^2 - \gamma v(v^2/2 - x^2 + x^3 - \mu)$$

where $\gamma > 0$ is a fixed parameter (cf. example 1.5). It is quite easy to show that the “energy” $H = v^2/2 - x^2 + x^3$ obeys $\dot{H} = -\gamma v^2(H - \mu)$, i.e. $H(x(t), v(t)) - \mu = (H(x(0), v(0)) - \mu) \exp(-\gamma \int_0^t v^2(t')dt')$. Therefore, as $t \to \infty$ either $v(t) \to 0$ or $H(x(t), v(t)) \to \mu$. The first case corresponds to one of the fixed point solutions $(0, 0)$ or $(2/3, 0)$, or to the stable manifold of the saddle point $(0, 0)$. The second possibility $H(x(t), v(t)) \to \mu$, which covers the vast majority of initial conditions depends on the value of the parameter $\mu$. If $\mu = \mu_0 = 0$ the level curve $H = 0$ yields a homoclinic orbit. For $-4/27 < \mu < \mu_0 = 0$ the condition $H = \mu$ determines a closed orbit, i.e., a periodic orbit is created at $\mu < \mu_0$. On the contrary no closed orbit obeys $H = \mu > 0$, i.e., the periodic orbit is destroyed in the homoclinic bifurcation.

Summary: by studying bifurcations, i.e. instabilities, we are able to reduce the number of degrees of freedom (to a centre manifold), we can understand how new orbits are created, and we can reconstruct the dynamics in large parts of the phase space.