Turing and the Riemann Hypothesis

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The Riemann zeta function

In 1859 Bernard Riemann studied the following function in connection with the distribution of prime numbers:

\[ \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]

Note the gap at \( s = 1 \) because the harmonic numbers

\[ H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \]

go to \( \infty \) as \( n \) goes to \( \infty \).
Analytic continuation

The sum $\sum_{n=1}^{\infty} \frac{1}{n^s}$ only makes sense for $s > 1$. But we can make a continuation to the whole complex plane (numbers of the form $a + b \times \sqrt{-1}$).

This is analogous to the continuation (extrapolation) from individual points to a continuous line:
The Riemann zeta function for complex numbers

One version of the continuation of zeta:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^n \binom{n}{k} \frac{1}{(1 + k)^s}.$$ 

Riemann conjectured that the only time this version of $\zeta(s)$ is zero is when

$$s = \frac{1}{2} + t\sqrt{-1},$$

for some value of $t$ (except for some trivial cases).

This is a line through $x = \frac{1}{2}$, parallel to the $y$ axis. It is called the critical line.

**The Riemann Hypothesis:** all nontrivial zeros of $\zeta(s)$ lie on the critical line.
Why does RH matter?

The **Prime Number Theorem** says that $\pi(x)$, the number of primes $\leq x$, is ‘approached’ by $\frac{x}{\ln x}$ (the ratio of this to $\pi(x)$ gets closer and closer to 1).

The actual *difference* between $\pi(x)$ and $x/\ln x$ gets bigger and bigger. All we know is, it is bounded by roughly some power of $x$:

$$\left| \pi(x) - \frac{x}{\ln x} \right| \leq Cx^{\alpha+1/2},$$

where $C$ is a constant.

What is $\alpha$? It is the maximum horizontal distance of zeros of $\zeta(s)$ from the critical line.

So RH $\Rightarrow \alpha = 0 \Rightarrow$ a square root error in our approximation of $\pi(x)$.

(Actually there is some small print which replaces the $x/\ln x$ approximation in all this with a more accurate ‘integral function’ called Li$(x)$; but the basic idea is the same.)
Jeffrey Lagarias’s version of RH

An elementary (not involving $\sqrt{-1}$) equivalent to RH was found by Jeffrey Lagarias in 2001, using a 1984 theorem of Guy Robin:

RH is true if and only if:

For all positive integers $n$,

$$\sigma(n) \leq H_n + e^{H_n} \ln H_n,$$

where $\sigma(n)$ is the sum of the positive divisors of $n$, not including $n$:

\[
\begin{align*}
\text{E.g. } \sigma(6) &= 1 + 2 + 3 + 6 = 12 \\
H_6 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 2.45 \\
2.45 + e^{2.45} \times \ln(2.45) &\approx 2.45 + 11.59 \times 0.90 \approx 12.88
\end{align*}
\]
Turing’s involvement in RH

Turing worked intermittently on calculating zeros of $\zeta(s)$. His last ever research paper (published 1953) proposed a method for calculating

$$N(t) = \text{no. of zeros } a + b \times \sqrt{-1} \text{ with } 0 < b \leq t,$$

(with $a = 1/2$ if RH is true).

At this time there was little empirical evidence for RH and Turing was a sceptic, looking for counterexamples.

Today over 10 billion zeros have been checked and found to lie on the critical line. But the ‘Turing method’ for $N(t)$ is still in use and has been extended to related parts of analytic number theory.