1. Suppose a graph \( G = (V, E) \), with \( |V| \geq 3 \), has the following properties:
   (a) For any two vertices \( x \) and \( y \) there is exactly one other vertex \( z \) which is adjacent to both \( x \) and \( y \);
   (b) There is some vertex \( v \) which is adjacent to all others.

Prove that \( G \) must consist of a collection of triangles joined at a common ‘hub’ vertex, as in the example on the right.

(The famous Friendship Theorem of Paul Erdős, Alfréd Rényi and Vera Sós says that, in fact, property (a) above implies property (b)ǃ See more here: www.theoremoftheday.org/Theorems.html#163)

**SOLUTION:** Let \( x \) be any vertex other than the hub vertex \( v \). Now \( x \) and \( v \) must share exactly one common ‘neighbour’ say, \( y \), to which they are both adjacent. Since \( v \) is adjacent to both \( x \) and \( y \) by (2) it is enough for there to be an edge joining \( x \) and \( y \). Thus \( x, y \) and \( v \) form a triangle. Moreover, neither \( x \) nor \( y \) can be adjacent to any other vertex. For suppose \( x \), say, was adjacent to \( z \neq y, v \). Then \( z \) is adjacent to \( v \) by (2), but then \( v \) and \( x \) share two common neighbours, in contradiction of (1). Hence the vertex set must be partitioned into triangles sharing a common vertex \( v \).

2. A non-increasing sequence of non-negative integers is a **degree sequence** if some graph \( G \) has these integers as its vertex degrees. For example, the sequence \((4, 3, 3, 3, 2, 1)\) is a degree sequence because these are precisely the vertex degrees of the graph shown on the right.

A 1960 theorem of Paul Erdős and Tibor Gallai says that a non-increasing sequence of non-negative integers \((d_1, d_2, \ldots, d_n)\) is a degree sequence if and only if \( \sum d_i \) is even and, for all \( k, 1 \leq k \leq n \),
\[
\sum_{i=1}^{k} d_i - \sum_{i=k+1}^{n} \min(d_i, k) \leq k(k-1).
\]

**(1)**

(a) Why do we require that \( \sum d_i \) is even?

(b) Confirm that the degree sequence \((4, 3, 3, 3, 2, 1)\) satisfies inequality (1).

(c) What is the complexity of testing this inequality for a sequence of length \( n \)?

**SOLUTION:**

(a) \( \sum d_i \) is even because the Handshaking Lemma says this is the case.

(b) The inequalities are as follows:

\[
\begin{align*}
1 & : 4 - \left( \min(3,1) + \min(3,1) + \min(3,1) + \min(2,1) + \min(1,1) \right) = 4 - (1 + 1 + 1 + 1 + 1) = -1 \leq 1 \times 0 = 0 \\
2 & : 4 + 3 - \left( \min(3,2) + \min(3,2) + \min(2,2) + \min(1,2) \right) = 7 - (2 + 2 + 2 + 1) = 0 \leq 2 \times 1 = 2 \\
3 & : 4 + 3 + 3 - \left( \min(3,3) + \min(2,3) + \min(1,3) \right) = 10 - (3 + 2 + 1) = 4 \leq 3 \times 2 = 6 \\
4 & : 4 + 3 + 3 + 3 - \left( \min(2,4) + \min(1,4) \right) = 13 - (2 + 1) = 10 \leq 4 \times 3 = 12 \\
5 & : 4 + 3 + 3 + 3 + 2 - \min(1,5) = 15 - 1 = 14 \leq 5 \times 4 = 20 \\
6 & : 4 + 3 + 3 + 3 + 2 + 1 = 16 \leq 6 \times 5 = 30
\end{align*}
\]
(c) We are investigating the complexity of an arithmetic calculation, so we do not count the 'number of steps' but instead the number of additions, min calculations and comparisons \((\leq)\).

For inequality \(k\) we have \(n\) additions (from \(\sum_{i=1}^{k}\) and \(\sum_{i=k+1}^{n}\)), \(n - k\) mins and 1 comparison.

So the total over \(k = 1, \ldots, n\) is \(n^2\) additions, \(n - 1 + n - 2 + n - 3 + \ldots + 2 + 1\) mins and \(n\) comparisons.

This gives a total of \(n^2 + \frac{1}{2}n(n - 1) + n\) arithmetic operations, which is \(O(n^2)\).

3. Dijkstra’s algorithm fails to work properly if negative edge-weights are allowed. Consider the following strategy which attempts to overcome this problem:

Let \(\omega\) be the negative weight of greatest absolute value. Add \(|\omega| + 1\) to every weight to create a graph with only positive edge weights. Run Dijkstra’s algorithm on the result. Now subtract \(|\omega| + 1\) from every edge of the resulting tree.

Does this strategy work?

**SOLUTION:** No! Look at the example on page 1 of Week 4, Lecture notes 1:

![Diagram](image)

The strategy dictates that we add 3 to every edge. But this means longer paths will have more multiples of 3 added to their total weight, cancelling out the effect of negative edges. The path \(ab, bc\) in the graph has weight \(3 - 2 = 1\) which is less than \(ac\) with weight 2. But adding 3 to all edges makes the comparison \((3 + 3) + (-2 + 3) = 1 + 6\) against \(2 + 3 = 5\). Again the path \(ac\) is chosen but this is not the shortest path in the original graph.

4. [FOR MARKING]

(a) Run Dijkstra’s algorithm on the following graph \(G\) using the root vertex \(v = a\). Show, diagrammatically or by means of a table, how the algorithm operates at each iteration.

Give the final tree and table of distances which are the outputs of the algorithm.

(b) Draw a minimum-weight spanning tree for the graph \(G\) in part (a). You need not show how you obtained this tree.

(c) Is your tree in part (a) a minimum-weight spanning tree for \(G\)? Does your tree in part (b) give minimum-weight paths from \(a\) to all other vertices? Give counterexamples where appropriate.

A jpeg version of the graph for this question, which you may care to reproduce in your answer, is available on the Week 4 page of the MTH6105 website.
SOLUTION:

(a) The diagrammatic version is:

Table of distances:

<table>
<thead>
<tr>
<th>x</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>d(a,x)</td>
<td>0</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

[3 marks for final tree + table]
The table version of the solutions is:  

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>u</th>
<th>tree edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>a</td>
<td>Ø</td>
</tr>
<tr>
<td>1</td>
<td>(a, 5)</td>
<td>(a, ∞)</td>
<td>(a, 1)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>d</td>
<td>{ad}</td>
</tr>
<tr>
<td>2</td>
<td>(a, 5)</td>
<td>(a, ∞)</td>
<td>(d, 1 + 2)</td>
<td>(a, ∞)</td>
<td>(d, 1 + 5)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>e</td>
<td>{ad, de}</td>
</tr>
<tr>
<td>3</td>
<td>(e, 3 + 1)</td>
<td>(a, ∞)</td>
<td>(e, 3 + 6)</td>
<td>(d, 6)</td>
<td>(e, 3 + 4)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>b</td>
<td>{ad, de, be}</td>
</tr>
<tr>
<td>4</td>
<td>(b, 4 + 1)</td>
<td>(e, 9)</td>
<td>(d, 6)</td>
<td>(e, 7)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>c</td>
<td>{ad, de, be, bc}</td>
</tr>
<tr>
<td>5</td>
<td>(c, 5 + 2)</td>
<td>(d, 6)</td>
<td>(e, 7)</td>
<td>(e, 7)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>g</td>
<td>{ad, de, bc, dg}</td>
</tr>
<tr>
<td>6</td>
<td>(c, 7)</td>
<td>(e, 7)</td>
<td>(e, 7)</td>
<td>(e, 7)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>f</td>
<td>{ad, de, bc, dg, cf}</td>
</tr>
<tr>
<td>7</td>
<td>(e, 7)</td>
<td>(f, 7 + 2)</td>
<td>(e, 7)</td>
<td>(f, 7 + 2)</td>
<td>(e, 7)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>h</td>
<td>{ad, de, bc, dg, cf}</td>
</tr>
<tr>
<td>8</td>
<td>(h, 7 + 1)</td>
<td>(e, 7)</td>
<td>(f, 7 + 2)</td>
<td>(e, 7)</td>
<td>(f, 7 + 2)</td>
<td>(e, 7)</td>
<td>(a, ∞)</td>
<td>(a, ∞)</td>
<td>i</td>
<td>{ad, de, bc, dg, cf}</td>
</tr>
</tbody>
</table>

(b) A minimum-weight spanning tree is as shown below:  

(c) The solution tree in part (a) is not a minimum-weight spanning tree since it has total weight 17 whereas the tree in part (b) has total weight 12.  

The minimum-weight spanning tree does not give minimum distances. For example, the distance from a to i in this tree is 9, whereas the tree in part (a) reveals that \( d_G(a, i) = 8 \).