Introduction to Representations of $GL(n)$

J.M. Selig
Faculty of Business, Computing and Information Management,
London South Bank University,
London SE1 0AA, UK

1 Introduction—the group $GL(n)$

The general linear group is the group of all $n \times n$ non-singular matrices. Notice that this is indeed a group; it satisfies the group axioms,

- The product of any two $n \times n$ non-singular matrices is another $n \times n$ non-singular matrix.
- Matrix multiplication is associative.
- The inverse element to any group element is given by the inverse matrix. This is always possible because the elements of $GL(n)$ are non-singular.
- The identity element of the group is the $n \times n$ identity matrix.

Further the group $GL(n)$ is a Lie group, this just means that the multiplication and the map which sends every element to its inverse are both differentiable maps. This is easy to see since the maps can be defined in terms of addition, multiplication and division by non-zero numbers.

We will usually take the ground field, the set containing the matrix entries, to be complex numbers. This is just for simplicity, it is sometimes more useful to know the case where the ground field is the real numbers or some finite field. It is possible to say something about these cases but as usual, when we use complex numbers the theory is more complete. We can write $GL(n, \mathbb{R})$ to distinguish the case where the ground field is the real numbers.

As a manifold the group $GL(n)$ will be an open set in the space $\mathbb{C}^{n^2}$, the complement of the closed set consisting of the singular matrices. So the group has complex dimension n^2, real dimension $2n^2$. Notice that the group $GL(n, \mathbb{R})$, real dimension n^2, has 2 disjoint components, a piece containing matrices with positive determinant and a bit with the negative determinant matrices. Further notice that the matrices with positive determinant form a subgroup of $GL(n, \mathbb{R})$, but the negative matrices do not.

2 Representations

A representation of a group is a linear action of a group on a vector space. To each group element g, we associate a linear mapping $L(g)$. This must be done in such a way that the mapping obeys,

$$L(g_1)L(g_2) = L(g_1 g_2), \qquad \text{for all } g_1,\, g_2 \in GL(n).$$

The $L(g)$s must be linear maps, that is,

$$L(g)(\mathbf{v}_1 + \mathbf{v}_2) = L(g)\mathbf{v}_1 + L(g)\mathbf{v}_2.$$

This means that once we have chosen a basis for the vector space, we can identify the linear maps with matrices. From the first condition above we can see that the identity element of the group e must be mapped to the identity matrix, $L(e) = I$. Also that inverse of a group element must be mapped to the inverse matrix, $L(g^{-1}) = L(g)^{-1}$. If the vector space is m dimensional then the invertible linear transformations are just $GL(m)$. Hence we could view a representation as a homomorphism from $GL(n)$ to $GL(m)$.

$$L : GL(n) \longrightarrow GL(m).$$

The dimension of such a representation would be m.

However, this is not the usual way to view representations, in part because it depends on a choice of basis for the vector space. Really we want to define the representation without having to specify a particular basis. One way to do this is to assume that the representation we get after a change of basis is equivalent to the original representation. For example, suppose that under a change of basis in the vector space all vector $\mathbf{v}$ become $\mathbf{u} = A\mathbf{v}$, where A is a non-singular $m \times m$ matrix. In this new basis our original representation L will become,

$$g \longmapsto AL(g)A^{-1}, \qquad \text{for all } g \in GL(n).$$

So equivalent representations are related by a similarity transformation.

In the next section we look at some example and try to explain why this concept is so useful.

3 Examples

3.1 The Standard rep.

Let V be an n-dimensional vector space. The set of $n \times n$ matrices act on this space. This is the standard representation of $GL(n)$, each group element is mapped to its matrix.

3.2 The Dual rep.

The dual of a vector space V, is the space of linear functionals on V. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for V then the dual V^* has a basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ such that,

$$\mathbf{f}_i(\mathbf{e}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

The evaluation map $\mathbf{f}_i(\mathbf{e}_j)$ can be thought of as the product of a row vector with a column vector, suppose $\mathbf{e}_i$ is an n-dimensional column vector with a 1 in row and zero's everywhere else, suppose $\mathbf{f}_j$ is similarly a column vector this time with the only non-zero entry in the j-th row, the evaluation can now be written,

$$\mathbf{f}_i(\mathbf{e}_j) = \mathbf{f}_j^T \mathbf{e}_i$$

Now if we transform the vectors in V according to $L(g)\mathbf{e}_i$ then to preserve the results of the evaluation the elements of the dual vector space V^* must transform according to $L(g)^{-T} f_j$, where the superscript $-T$ denotes the transpose of the inverse. This is the dual representation of the group. Notice that the (ij) entry in $L(g)^{-T}$ will be the (ij) minor of $L(g)$ divided by its determinant.

3.3 The Determinant rep.

Consider the map $\Delta : g \mapsto \det(g)$. This is a 1-dimensional representation of $GL(n)$. Remember that $\det(g_1 g_2) = \det(g_1)\det(g_2)$.

In fact there are several 1-dimensional representations, for any positive integer p we have a representation,

$$\Delta^P : g \longmapsto \det(g)^p.$$

When $p = 0$, the representation is called the trivial representation, we will write this as, $\mathbb{1} : g \mapsto 1$.

3.4 Linear subspaces—exterior powers

Let V be an n-dimensional vector space as usual, but assume that $n \geqslant 3$. Now consider the 2-dimensional vector subspaces of V. A 2-dimensional subspace is defined by any pair of linearly independent vectors. But of course may pairs of vectors define the same plane. It is not too difficult to see that two pairs of vectors $\mathbf{x}$, $\mathbf{y}$ and $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, define the same plane in V if and only if they are related by,

$$\tilde{\mathbf{x}} = a\mathbf{x} + b\mathbf{y}, \qquad \tilde{\mathbf{y}} = c\mathbf{x} + d\mathbf{y}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$, that is to say $ad - bc \neq 0$. These vectors can be written in terms of the standard basis of V as, for example, $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$. So in terms of these components the transformation above can be written,

$$\begin{pmatrix} \tilde{x}_i \\ \tilde{y}_i \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad i = 1, \dots n.$$

This can be extended for pairs of components to,

$$\begin{pmatrix} \tilde{x}_i & \tilde{x}_j \\ \tilde{y}_i & \tilde{y}_j \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}, \qquad 1 \leqslant i < j \leqslant n.$$

Now if we take the determinant of both sides we get

$$\det \begin{pmatrix} \tilde{x}_i & \tilde{x}_j \\ \tilde{y}_i & \tilde{y}_j \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}, \qquad 1 \leqslant i < j \leqslant n.$$

Notice that for any pair of indices ij, the transformation just multiplies the quantity $\det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$ by a constant factor; the determinant of the transformation matrix $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This means we can use the quantities $\det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$, as coordinates in a projective space. If fact these are the well known Plücker coordinates,

$$p_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}, \qquad 1 \leqslant i < j \leqslant n.$$

Two different planes will have different Plücker coordinates (unless the difference is just multiplication by a constant). However, not all sets of Plücker coordinates represent planes, only those that satisfy the "Plücker relations". This leads us towards the geometry of Grassmann manifolds, the manifold whose points are k-planes in an n-dimensional vector space.

However, if we stay with the vector space defined by the Plücker coordinates, then we can ask how does the group $GL(n)$ act on this space? Let us write an arbitrary element of $GL(n)$ as an $n \times n$ matrix with entries $A = (a_{ij})$. The effect of this on a vector $\mathbf{x}$ will be,

$$\mathbf{x}' = A\mathbf{x},$$

or in terms of the components of the vector,

$$x_i' = \sum_{k=1}^{n} a_{ik} x_k, \qquad i = 1, \dots, n.$$

The corresponding action on the Plücker coordinates is given by,

$$\begin{aligned} p_{ij}' &= \det \begin{pmatrix} x_i' & x_j' \\ y_i' & y_j' \end{pmatrix} \\ &= \det \begin{pmatrix} a_{i1}x_1 + \cdots + a_{in}x_n & a_{j1}x_1 + \cdots + a_{jn}x_n \\ a_{i1}y_1 + \cdots + a_{in}y_n & a_{j1}y_1 + \cdots + a_{jn}y_n \end{pmatrix} \\ &= \sum_{k,l=1}^{n} \det \begin{pmatrix} a_{ik}x_k & a_{il}x_l \\ a_{ik}y_k & a_{il}y_l \end{pmatrix} \\ &= \sum_{k,l=1}^{n} a_{ik}a_{il} \det \begin{pmatrix} x_k & x_l \\ y_k & y_l \end{pmatrix} \\ &= \sum_{1 \leqslant k < l \leqslant n} (a_{ik}a_{il} - a_{il}a_{ik}) \det \begin{pmatrix} x_k & x_l \\ y_k & y_l \end{pmatrix} \end{aligned}$$

$$= \sum_{1 \leqslant k < l \leqslant n} (a_{ik}a_{il} - a_{il}a_{ik})p_{kl}.$$

This means that if we arrange the Plücker coordinates in a vector then the transformation can be represented by a matrix whose elements are the 2×2 minors of the original transformation A. This representation of $GL(n)$ is called the antisymmetric, or exterior, square of the standard representation, written $\bigwedge^2 GL(n)$.

All this generalises quite easily, for 3-dimensional subspaces the Plücker coordinates are given by,

$$p_{ijk} = \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix},$$

based on any three independent vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in the 3-space. The representation of the group $GL(n)$ on these Plücker coordinates is given by the matrix 3×3 minors of the original transformation A, this is $\bigwedge^3 GL(n)$ the exterior cube of the standard representation.

As long as $k < n$ we can form the kth exterior power of the standard representation by taking the matrix of $k \times k$ minors of A. The representation $\bigwedge^n GL(n)$ is the determinant representation we met in the previous subsection.

3.5 Polynomials—symmetric powers

Consider the space of all degree 2 homogeneous polynomials in 3 variables. A typical polynomial here will have the form,

$$p(x_1, x_2, x_3) = ax_1^2 + 2bx_1x_2 + 2cx_1x_3 + dx_2^2 + 2ex_2x_3 + fx_3^2,$$

this can be written rather neatly as a product of matrices,

$$p(x_1, x_2, x_3) = \begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The variables here x_1, x_2 and x_3 can be thought of as homogeneous coordinates in a 2-dimensional projective space. The zero set of such a polynomial will be a conic curve.

Now suppose we change coordinates, how should the coefficients $a, \dots, f$ be transformed in order that the conic curve is preserved? As usual we assume that the transformation on coordinates is given by an element of $GL(n)$ (in this case $n = 3$), that is $\mathbf{x}' = A\mathbf{x}$ or in vector form,

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

or in component form,

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad i = 1,\ 2,\ 3.$$

If we write the coefficient matrix as,

$$B = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

then we can see that it must transform according to,

$$B' = A^{-T} B A^{-1}.$$

In older language this was called a substitution.

If the conic curve were actually a pair of straight lines, its equation would be,

$$(f_1 x_1 + f_2 x_2 + f_3 x_3)(h_1 x_1 + h_2 x_2 + h_3 x_3) = 0.$$

In this case the symmetric coefficient matrix would be,

$$B = \frac{1}{2} \left\{ \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \begin{pmatrix} h_1, & h_2, & h_3 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \begin{pmatrix} f_1, & f_2, & f_3 \end{pmatrix} \right\}.$$

The vectors $\mathbf{f}$ and $\mathbf{h}$ clearly transform according to the dual representation we met above and hence we say that the matrix B transforms according to the symmetric square of the dual representation. This representation can be denoted $\text{Sym}^2 GL(3)^*$. The asterisk is used here to denote the dual representation.

Clearly, we may extend these ideas into higher dimensions and higher degree polynomials. However, the nice link with symmetric matrices disappears and we have to use tensors. For example, for cubic polynomials we would replace the coefficient matrix with a degree three symmetric tensor b_{ijk}, a general cubic polynomial is then given by,

$$p(\mathbf{x}) = \sum_{ijk} b_{ijk} x_i x_j x_k.$$

Under a coordinate transformation the element of the tensor transform according to the representation $\text{Sym}^3 GL(n)^*$, that is the new elements will be given by,

$$b'_{ijk} = \sum_{lmn} b_{lmn} A_{li} A_{mj} A_{nk},$$

where A_{li} is an element of the dual representation of the standard rep.

4 Operations on Representations

In this section we look at two standard ways to combine a pair of representations into a third rep.

Suppose $L(g)$ and $M(g)$ are two representations of $GL(n)$. Then we can form their direct sum $L \oplus M\ (g)$. If L is a k-dimensional representation and M is l-dimensional, then their direct sum will be $(k+l)$-dimensional. Matrices from the direct sum will have the general form,

$$L \oplus M\,(g) = \begin{pmatrix} L(g) & 0 \\ 0 & M(g) \end{pmatrix}.$$

Notice that $L \oplus M$ and $M \oplus L$ are different representations but it is easy to see that these representations are equivalent.

Suppose that the vector space that $L(g)$ acts on has a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ and the space that $M(g)$ acts on has a basis $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_l\}$, then the tensor product representation acts on a space with basis $\{\ldots, \mathbf{e}_j \otimes \boldsymbol{\epsilon}_j, \ldots\}$. The action of the tensor product representation on this basis is as follows,

$$L \otimes M\,(g)\big(\mathbf{e}_j \otimes \boldsymbol{\epsilon}_j\big) = L(g)\mathbf{e}_j \otimes M(g)\boldsymbol{\epsilon}_j.$$

Remember that,

$$(a\mathbf{e}_i + b\mathbf{e}_j) \otimes (c\boldsymbol{\epsilon}_k + d\boldsymbol{\epsilon}_l) = ac(\mathbf{e}_i \otimes \boldsymbol{\epsilon}_k) + ad(\mathbf{e}_i \otimes \boldsymbol{\epsilon}_l) + bc(\mathbf{e}_j \otimes \boldsymbol{\epsilon}_k) + bd(\mathbf{e}_j \otimes \boldsymbol{\epsilon}_l)$$

The symmetric and antisymmetric powers of representations that we met above can be written in terms of the tensor powers of representation. The basis of an antisymmetric power of an n-dimensional representation L, say $\bigwedge^k L$ are,

$$\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k} = \frac{1}{k!} \sum_{\pi} \mathrm{sign}(\pi) \mathbf{e}_{\pi(i_1)} \otimes \mathbf{e}_{\pi(i_2)} \otimes \cdots \otimes \mathbf{e}_{\pi(i_k)},$$

where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and π ranges over all permutations of the k indices $i_1, \ldots i_k$.

Likewise the basis of the symmetric power $\mathrm{Sym}^k L$ will can be thought of as,

$$\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_k} = \frac{1}{k!} \sum_{\pi} \mathbf{e}_{\pi(i_1)} \otimes \mathbf{e}_{\pi(i_2)} \otimes \cdots \otimes \mathbf{e}_{\pi(i_k)},$$

here though, we can allow the indices to coincide.

As a small example we look at the symmetric and antisymmetric squares of a 3-dimensional representation. The basis vectors for the antisymmetric square will be,

$$\mathbf{e}_1 \wedge \mathbf{e}_2 = \frac{1}{2}(\mathbf{e}_1 \oplus \mathbf{e}_2 - \mathbf{e}_2 \oplus \mathbf{e}_1), \quad \mathbf{e}_1 \wedge \mathbf{e}_3 = \frac{1}{2}(\mathbf{e}_1 \oplus \mathbf{e}_3 - \mathbf{e}_3 \oplus \mathbf{e}_1), \quad \mathbf{e}_2 \wedge \mathbf{e}_3 = \frac{1}{2}(\mathbf{e}_2 \oplus \mathbf{e}_3 - \mathbf{e}_2 \oplus \mathbf{e}_3).$$

For the symmetric square we get basis elements,

$$\mathbf{e}_1^2 = \mathbf{e}_1 \oplus \mathbf{e}_1, \quad \mathbf{e}_2^2 = \mathbf{e}_2 \oplus \mathbf{e}_2, \quad \mathbf{e}_3^2 = \mathbf{e}_3 \oplus \mathbf{e}_3,$$

and

$$\mathbf{e}_1 \mathbf{e}_2 = \frac{1}{2}(\mathbf{e}_1 \oplus \mathbf{e}_2 + \mathbf{e}_2 \oplus \mathbf{e}_1), \quad \mathbf{e}_1 \mathbf{e}_3 = \frac{1}{2}(\mathbf{e}_1 \oplus \mathbf{e}_3 + \mathbf{e}_3 \oplus \mathbf{e}_1), \quad \mathbf{e}_2 \mathbf{e}_3 = \frac{1}{2}(\mathbf{e}_2 \oplus \mathbf{e}_3 + \mathbf{e}_2 \oplus \mathbf{e}_3).$$

Finally here we remark that when these operations are applied to several representations they obey some of the familiar laws of arithmetic, they are associative,

$$L \oplus (M \oplus N) = (L \oplus M) \oplus N, \qquad L \otimes (M \otimes N) = (L \otimes M) \otimes N.$$

Also the tensor product distributes over the direct sum,

$$L \otimes (M \oplus N) = (L \otimes M) \oplus (L \otimes N).$$

Moreover, the trivial representation, the one which sends every group element to the number 1, acts as a unit for tensor product. We can write this as,

$$L \otimes \mathbb{1} = \mathbb{1} \otimes L = L.$$

So the set of all (equivalence classes of) representations of $GL(n)$ for some particular n, is very nearly a ring with $\oplus$ and $\otimes$ as the ring operations. The problem is that the representations do not form an abelian group under $\oplus$ because there aren't any inverses to representations under $\oplus$.

5 Characters

The character of a representation is essentially the trace of the matrices. To be precise the character is a map from the group to the complex numbers which depends on the representation, $\chi_L : GL(n) \to \mathbb{C}$. The map is given by,

$$\chi_L(g) = \mathrm{Tr}(L(g)).$$

Now any $g \in GL(n)$ can be put into Jordan normal form by a suitable similarity transformation hgh^{-1}. The trace of a matrix in Jordan normal form is simply the sum of its eigenvalues. The trace has the property that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ so that,

$$\mathrm{Tr}(hgh^{-1}) = \mathrm{Tr}(h^{-1}hg) = \mathrm{Tr}(g).$$

So the character of the standard representation simply maps each g to the sum of its eigenvalues $(t_1 + t_2 + \cdots + t_n)$ say.

In fact we can see that for any representation the character only depends on the eigenvalues of g,

$$\mathrm{Tr}\left(L(hgh^{-1}\right) = \mathrm{Tr}\left(L(h)L(g)L(h)^{-1}\right) = \mathrm{Tr}\left(L(h)^{-1}L(h)L(g)\right) = \mathrm{Tr}\left(L(g)\right).$$

From now on we will only deal with polynomial representations, these are representation where the matrix entries are polynomial functions of the matrix entries of the original g. In fact all the representations we have seen so far, except one, are polynomial representation. The exception is the dual representation, this is a rational representation, its matrix entries are the ratios of polynomial functions in the original matrix entries. However, it is easy to see that if we take the tensor product of the dual rep. with the determinant rep. Δ, we get the antic symmetric power representation $\bigwedge^{n-1}$. In fact we can turn any rational representation into a polynomial representation by tensoring with some power of the determinant rep.

Clearly, the character of a polynomial representation will be a polynomial in $t_1,\ t_2, \ldots, t_n$ the eigenvalues of g. Moreover, since the permutation matrices are a discrete subgroup of $GL(n)$, the character will be a symmetric polynomial in the eigenvalues of g.

For example the characters of the antisymmetric powers are the elementary symmetric polynomials:

$$\begin{aligned}
\chi_{\bigwedge^1} &= t_1 + t_2 + \cdots + t_n \\
\chi_{\bigwedge^2} &= \sum_{1 \leqslant i < j \leqslant n} t_i t_j \\
\chi_{\bigwedge^3} &= \sum_{1 \leqslant i < j < k \leqslant n} t_i t_j t_k \\
&\vdots \\
\chi_{\bigwedge^n} &= t_1 t_2 \cdots t_n
\end{aligned}$$

Here the standard rep. has been written $\bigwedge^1$ and the determinant rep. $\bigwedge^n$.

The character behaves nicely with respect to our operations on representations. It is easy to see that,

$$\chi_{L \oplus M}(g) = \chi_L(g) + \chi_M(g),$$

for any representations L and M. And it is not much harder to see that,

$$\chi_{L \otimes M}(g) = \chi_L(g)\chi_M(g).$$

It is well known that the elementary symmetric polynomial generate the ring of all symmetric polynomials. So the relations above almost give a ring isomorphism between the representations of $GL(n)$ and the symmetric polynomials in n variables. "Almost" because the representations do not form a ring, there are no inverses for the direct sum operation. We can remedy this using the "Grothendick trick", by taking pairs of representation (L, M) and quotienting by the relation,

$$(L_1, L_2) \sim (M_1, M_2) \quad \Leftrightarrow \quad L_1 \oplus M_2 = L_2 \oplus M_1$$

(This trick can be used to produce integers from natural numbers or produce the rationals from the integers.)

The identity element with respect to the direct sum is the equivalence class of pairs of the form $[(L, L)]$. The inverse of an element $[(L, M)]$ is the element $[(M, L)]$.

The equivalence classes of these pairs now form a ring called the representation ring of the group and it is possible to show that this ring is isomorphic to the ring of symmetric polynomials.

The standard theory of representations of $GL(n)$ would now look at how to characterise irreducible representations and then how tensor products of these representations decompose as the direct sum of irreducibles.

Rather than follow that path here we look briefly at another important application.

6 Classical Invariant Theory

A classical invariant would be called a relative invariant in modern language. Such an invariant is just a one dimensional representation, a power of the determinant rep. An absolute invariant would be a copy of the trivial representation.

Recall from above that $\text{Sym}^2 GL(3)^*$ represents the coefficients of a conic curve in two dimensional projective space. Now the elements of the representation,$\text{Sym}^k\text{Sym}^2 GL(3)^*$ will be degree k polynomials in the coefficients. In general this representation will split into a number of irreducible representations,

$$\text{Sym}^k\text{Sym}^2 GL(3)^* = L \oplus M \oplus \cdots \oplus N.$$

If any of these sub-representations is isomorphic to Δ^p then it corresponds to a degree k polynomial in the coefficients of the conic which after a general linear substitution is simply multiplied by some power of the determinant. The power of the determinant is called the weight of the (relative) invariant. An invariant of weight 0 would be an absolute invariant.

For example, the discriminant of the curve,

$$\det \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

is clearly an invariant polynomial with degree 3 and weight -2. The invariants have geometric significance, the discriminant above vanishes precisely when the conic degenerates into a pair of lines.

If we could work out the character of $\text{Sym}^k\text{Sym}^2 GL(3)^*$ the invariant would appears as factors of the form $t_1^i t_2^i t_3^i$.

In fact known that all invariants of $\text{Sym}^2 GL(n)^*$ generated by the discriminant.

References

This work is taken mainly from section 4.1 of:

[1] Bernd Sturmfels. *Algorithms and Invariant theory*. Springer Verlag, Wein 1993.