On Hamiltonian cycle systems with a nice automorphism group

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15 April 2015

Hamiltonian cycle systems

- a Hamiltonian cycle system for the graph $\Gamma$, $|V(\Gamma)| = n$ is a set $B = \{C_1, \ldots, C_s\}$ of $n$-cycles of $\Gamma$
- such that the edges $E(C_1), \ldots, E(C_s)$ form a partition of $E(\Gamma)$
- often $\Gamma = K_n$, $n$ odd

![Graphs showing Hamiltonian cycles](image-url)
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more generally: graph decomposition

- Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$
- A decomposition of $G$ is a set of subgraphs of $G$ whose edge sets partition the edge set of $G$
- A $(G, \Gamma)$-decomposition is a decomposition in which the subgraphs are all isomorphic to $\Gamma$

A $(K_7, K_3)$-decomposition [or a $(K_7, C_3)$-decomposition]
the vertices of the $K_3$s are
$\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{0, 4, 5\}, \{1, 5, 6\}, \{0, 2, 6\}$
existence - Walecki’s construction

$n$ odd

$n$ even

removed 1-factor: $[0, 3], [1, 4], [2, 5], [\infty, \infty]$

regular cycle systems

- a HCS is **regular** if there is an automorphism group $G$ of $\Gamma$
  - acting sharply transitively on the vertices of $\Gamma$
    (so we identify $V(\Gamma)$ with $G$)
  - permuting the cycles of $B$
- it is called **cyclic** if $G$ is a cyclic group
- we shall call it **dihedral** if $G$ is a dihedral group
- to construct regular cs it is enough to give its **base cycles** –
  i.e. a set $\mathcal{F}$ of representatives for the $G$-orbits of the cycles
the existence of Hamiltonian cyclic CS is completely settled

- $n$ odd: $\exists$ a cyclic cs for $K_n$ iff $n \neq 15$ and $n \neq p^\alpha$ ($p$ an odd prime and $\alpha > 1$) [Buratti, Del Fra (2004)]
- $n$ even, $\exists$ a cyclic cs for $K_n - I$, iff $n \equiv 2, 4 \pmod{8}$ and $n \neq 2p^\alpha$ ($p$ an odd prime and $\alpha \geq 1$) [Gavlas-Jordon, Morris (2008)]
• we may consider a decomposition of $\Gamma$ into $k$-cycles ($k < n$)
• obvious necessary conditions for existence of a $k$-CS of $\Gamma$
  • $3 \leq k \leq V(\Gamma)$; the vertices of $\Gamma$ have even degree; $k \mid E(\Gamma)$
• when $\Gamma = K_n$, $n$ odd or $K_n - I$, $I$ a 1-factor, $n$ even these conditions are also sufficient
• Alspach and Gavlas-Jordon (2001) for $n$ and $k$ both odd or both even, Šajna (2002) in the remaining cases
• but for cyclic $k$-CS $\exists$ known only for $n \equiv 1, k \pmod{2k}$

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**dihedral HCS**

• consider a hamiltonian cycle system regular under the dihedral group $D_n$, $n$ even, $n = 2m$
• the graph is $K_{2m} - I$, $I$ a 1-factor

**Theorem (M. Buratti, FM, 2013)**

*There is a dihedral $(K_{2m} - I, C_{2m})$-design for all even $m$. There is a dihedral hamiltonian cycle system for $K_{2m} - I$, $m$ an odd integer, iff*

1. $m$ has at least two distinct prime factors
2. there is a suitable $e$ such that $p \equiv 1 \pmod{2^e}$ for all prime factors $p$ of $m$ and the number of those (counted with their respective multiplicities) such that $p \not\equiv 1 \pmod{2^{e+1}}$ is even.

• note that, in the cyclic case, there is no hamiltonian cycle system when $n \equiv 0 \pmod{8}$
• the odd integers $< 100$ for which there is a dihedral HCS are 21, 33, 45, 57, 65, 69, 77, 93
a dihedral HCS for $K_8 - I$

$D_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}, \ (x^3 = y^2 = 1, yxy = x^3)$

the removed 1-factor is $[1, y] \ [x^2, x^2y] \ [x^3, xy] \ [x, x^3y]$

sharply vertex-transitive HCS

- a sharply vertex-transitive HCS($n$) exists
  - for $n$ odd, iff $15 \neq n \neq p^\alpha \ p$ prime, $\alpha > 1$
  - for $n$ even, iff $15 \neq \frac{n}{2} \neq p^\alpha \ p$ prime, $\alpha \geq 1$
- this comes from the cyclic and dihedral results
- plus some extra cases when $\frac{n}{2} \equiv 3 \ (\text{mod} \ 4)$
  (Buratti, unpublished)
- but - for which groups $G$ is there a HCS which is sharply vertex transitive under $G$?
- not known and hard
1-rotational HCS\(n=2k+1\)

- symmetric terrace of a binary (1 involution, \(\lambda\)) group
- an ordering of the elts of \(G\) of the form
  \[(g_1, g_2, \ldots, g_k, g_k + \lambda, \ldots, g_2 + \lambda, g_1 + \lambda)\]
such that \(\{\pm(g_{i+1} - g_i, 1 \leq i \leq k - 1)\} = G \setminus \{0, \lambda\}\)
- e.g. for \(\mathbb{Z}_8\) \((0, 1, 7, 2, 6, 3, 5, 4)\)
- Bailey-Ollis-Preece (2003) ST \(\Rightarrow\) 1-rotational HCS under \(G\)
- Anderson-Ihrig (1993) \(G \neq Q_8\) and soluble \(\Rightarrow\) \(G\) has ST
- \(G\) non soluble - open
- Buratti-Rinaldi-Traetta (2013) 1-rotational HCS under \(G\)
  \(\Rightarrow\) ST
- BOP+BRT for \(k \geq 6\) up to isomorphism at least \(2^{\frac{3k}{4}}\)
  1-rotational \(HCS(2k+1)\)

**full automorphism group**

- want to determine the groups \(G\) for which there is an HCS whose full automorphism group is \(G\)
- there is a very recent result by Buratti, Lovegrove, Traetta
- necessary condition:
  - \(G \simeq AGL(1, p)\) \(p\) prime or
  - \(G\) is binary or
  - \(|G|\) is odd
- also sufficient with the possible exception of \(G\) binary and non soluble.