On Hamiltonian cycle systems with a nice automorphism group

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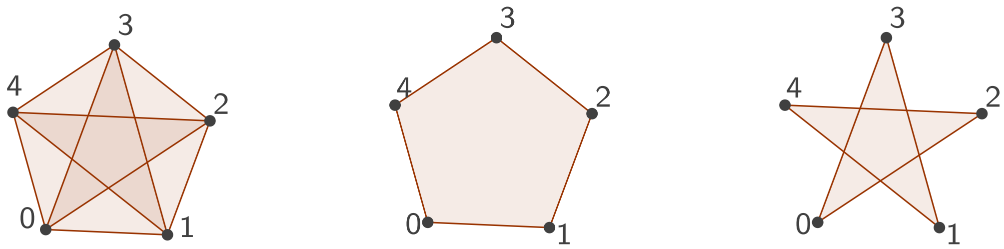
15 April 2015

Hamiltonian cycle systems

- a Hamiltonian cycle system for the graph Γ, $|V(\Gamma)| = n$ is a set $\mathcal{B} = \{C_1, \dots, C_s\}$ of n-cycles of Γ
- such that the edges $E(C_1), \dots E(C_s)$ form a partition of $E(\Gamma)$
- often $\Gamma = K_n$, n odd



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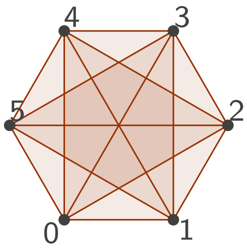

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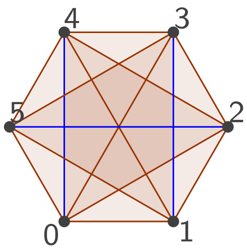

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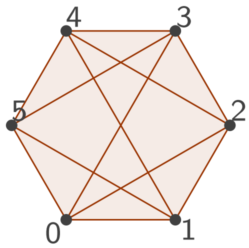

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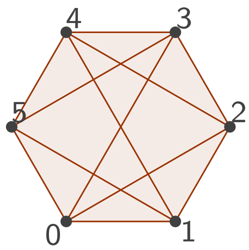
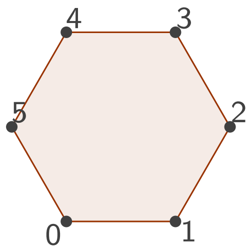
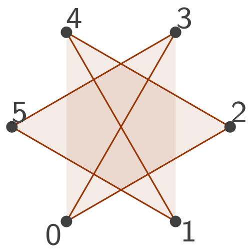
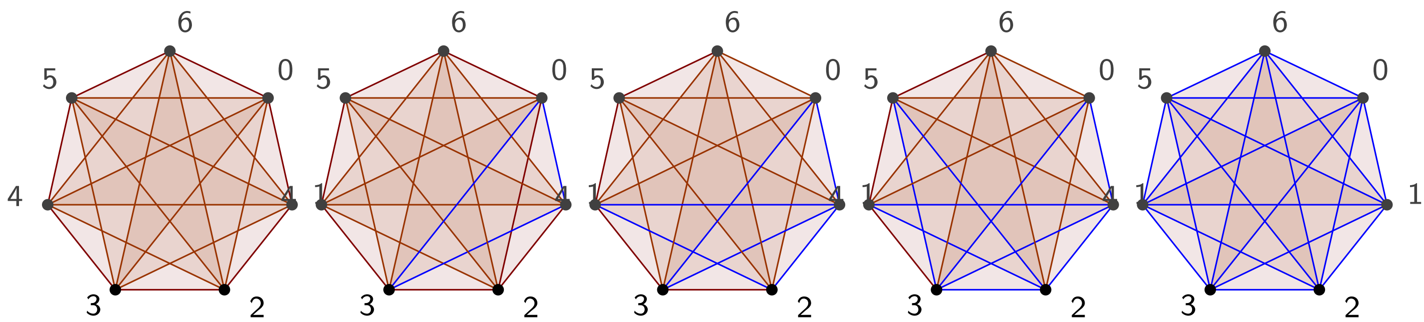






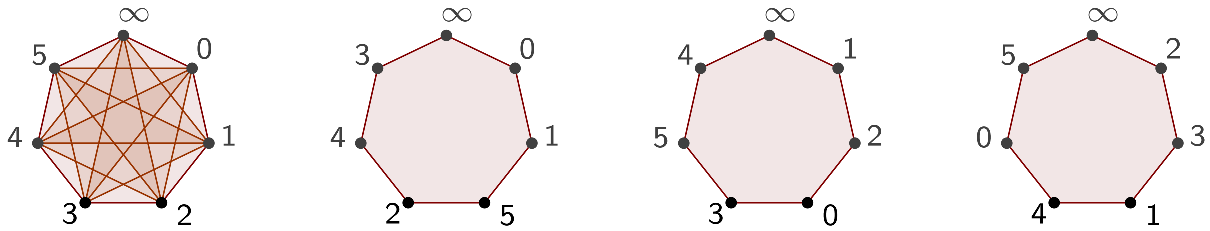
- Let G be a graph with vertex set $V(G)$ and edge set $E(G)$
- a decomposition of G is a set of subgraphs of G whose edge sets partition the edge set of G
- a (G, Γ)-decomposition is a decomposition in which the subgraphs are all isomorphic to Γ



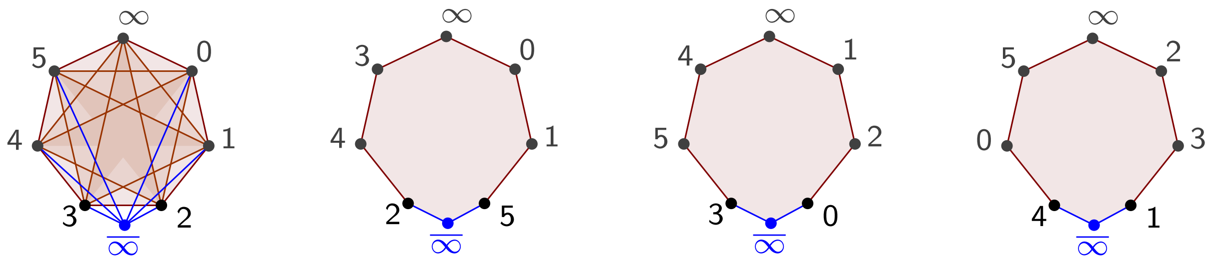
A (K_7, K_3)-decomposition [or a (K_7, C_3)-decomposition]
the vertices of the K_3s are
$\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{0,4,5\},\{1,5,6\},\{0,2,6\}$

existence - Walecki's construction

n odd



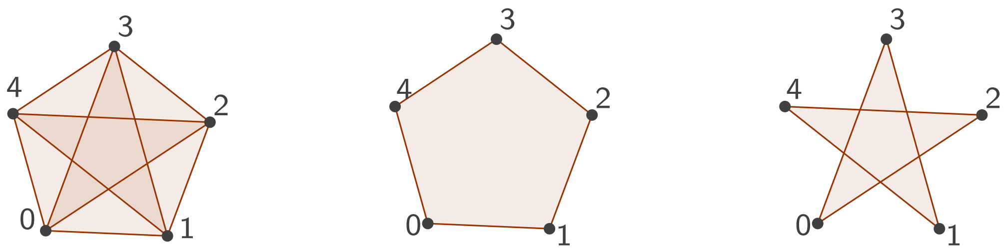
n even



removed 1-factor: $[0, 3]$, $[1, 4]$, $[2, 5]$, $[\infty, \overline{\infty}]$

regular cycle systems

- a HCS is regular if there is an automorphism group G of Γ
  - acting sharply transitively on the vertices of Γ (so we identify $V(\Gamma)$ with G)
  - permuting the cycles of $\mathcal{B}$
- it is called cyclic if G is a cyclic group
- we shall call it dihedral if G is a dihedral group
- to construct regular cs it is enough to give its base cycles – i.e. a set $\mathcal{F}$ of representatives for the G-orbits of the cycles

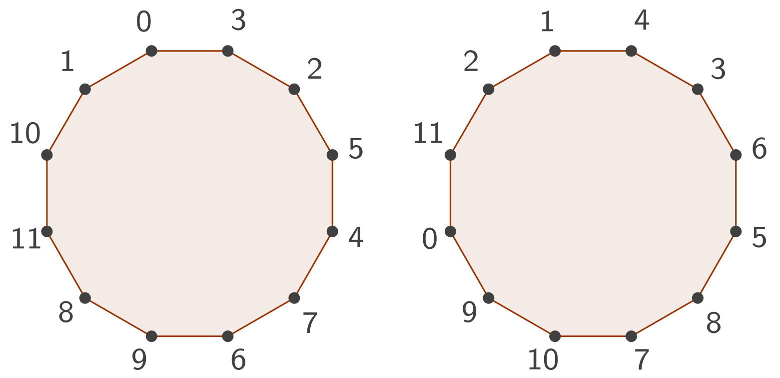


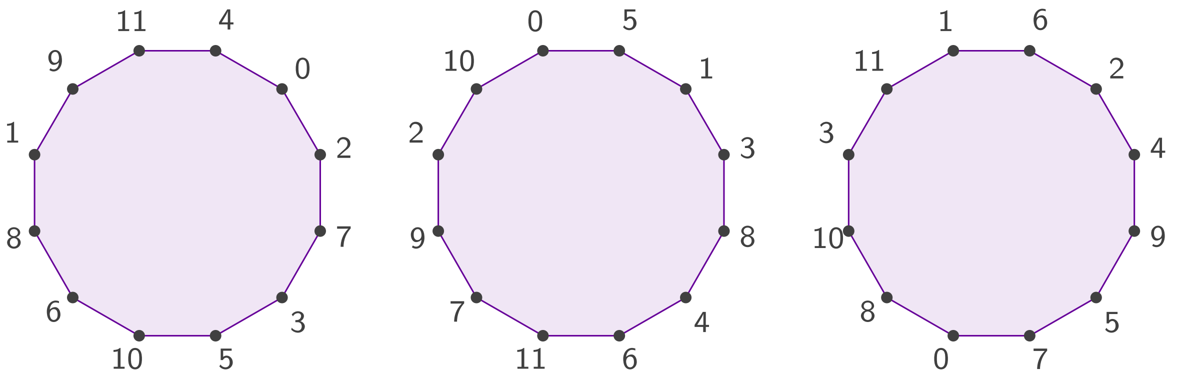
cyclic HCS

- the existence of hamiltonian cyclic CS is completely settled
- n odd: $\exists$ a cyclic cs for K_n iff $n \neq 15$ and $n \neq p^{\alpha}$ (p an odd prime and $\alpha > 1$) [Buratti, Del Fra (2004)]
- n even, $\exists$ a cyclic cs for $K_n - I$, iff $n \equiv 2, 4 \pmod 8$ and $n \neq 2p^{\alpha}$ (p an odd prime and $\alpha \geq 1$) [Gavlas-Jordon, Morris (2008)]

a cyclic HCS for $K_{12} - I$

removed 1-factor:
$[0, 6]$ $[1, 7]$ $[2, 8]$
$[3, 9]$ $[4, 10]$ $[5, 11]$





non-hamiltonian CS

- we may consider a decomposition of Γ into k-cycles ($k < n$)
- obvious necessary conditions for existence of a k-CS of Γ
  - $3 \leq k \leq V(\Gamma)$; the vertices of Γ have even degree; $k|E(\Gamma)$
- when $\Gamma = K_n$, n odd or $K_n - I$, I a 1-factor, n even these conditions are also sufficient
- Alspach and Gavlas-Jordon (2001) for n and k both odd or both even, Šajna (2002) in the remaining cases
- but for cyclic k-CS $\exists$ known only for $n \equiv 1, k \pmod{2k}$

dihedral HCS

- consider a hamiltonian cycle system regular under the dihedral group D_n n even, $n = 2m$
- the graph is $K_{2m} - I$, I a 1-factor

Theorem (M. Buratti, FM, 2013)

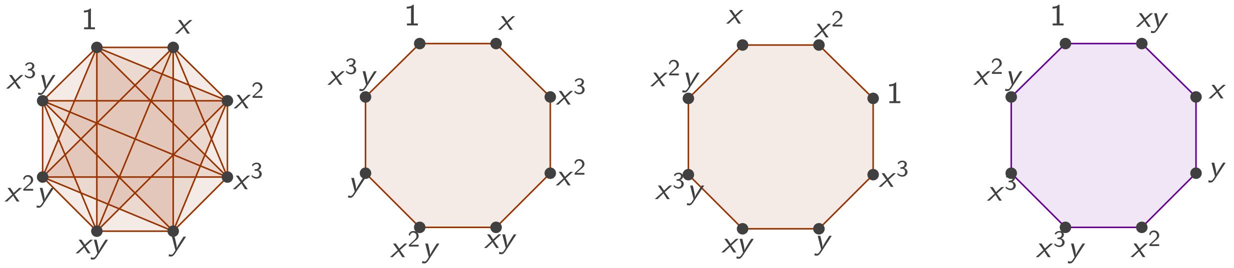
There is a dihedral $(K_{2m} - I, C_{2m})$-design for all even m.
There is a dihedral hamiltonian cycle system for $K_{2m} - I$, m an odd integer, iff

1. *m has at least two distinct prime factors*
2. *there is a suitable e such that $p \equiv 1 \pmod{2^e}$ for all prime factors p of m and the number of those (counted with their respective multiplicities) such that $p \not\equiv 1 \pmod{2^{e+1}}$ is even.*

- note that, in the cyclic case, there is no hamiltonian cycle system when $n \equiv 0 \pmod 8$
- the odd integers < 100 for which there is a dihedral HCS are $21, 33, 45, 57, 65, 69, 77, 93$

a dihedral HCS for $K_8 - I$

$D_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$, $(x^3 = y^2 = 1, yxy = x^3)$



the removed 1-factor is $[1, y]$ $[x^2, x^2y]$ $[x^3, xy]$ $[x, x^3y]$

sharply vertex-transitive HCS

- a sharply vertex-transitive $HCS(n)$ exists
  - for n odd, iff $15 \neq n \neq p^\alpha$ p prime, $\alpha > 1$
  - for n even, iff $15 \neq \frac{n}{2} \neq p^\alpha$ p prime, $\alpha \geq 1$
- this comes from the cyclic and dihedral results
- plus some extra cases when $\frac{n}{2} \equiv 3 \pmod 4$ (Buratti, unpublished)
- but - for which groups G is there a HCS which is sharply vertex transitive under G?
- not known and hard

1-rotational HCS(n=2k+1)

- symmetric terrace of a binary (1 involution, λ) group
- an ordering of the elts of G of the form

$$(g_1, g_2, \dots, g_k, g_k + \lambda, \dots, g_2 + \lambda, g_1 + \lambda)$$

  such that $\{\pm(g_{i+1} - g_i, 1 \leq i \leq k-1)\} = G \setminus \{0, \lambda\}$
- e.g. for $\mathbb{Z}_8$ $(0, 1, 7, 2, 6, 3, 5, 4)$
- Bailey-Ollis-Preece (2003) ST $\Rightarrow$ 1-rotational HCS under G
- Anderson-Ihrig (1993) $G \neq Q_8$ and soluble $\Rightarrow$ G has ST
- G non soluble - open
- Buratti-Rinaldi-Traetta (2013) 1-rotational HCS under G $\Rightarrow$ST
- BOP+BRT for $k \geq 6$ up to isomorphism at least $2^{\frac{3k}{4}}$ 1-rotational $HCS(2k+1)$

full automorphism group

- want to determine the groups G for which there is an HCS whose full automorphism group is G
- there is a very recent result by Buratti, Lovegrove, Traetta
- necessary condition:
  - $G \simeq AGL(1, p)$ p prime or
  - G is binary or
  - $|G|$ is odd
- also sufficient with the possible exception of G binary and non soluble.