Sylvester's Catalecticant

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1 Introduction

Work prompted by problem introduced by Tony, involved determinant of the form,

$$\det\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}.$$

After searching on Wikipedia this turns out to be an invariant discovered by Sylvester in 1852, [1]. The paper that introduces this determinant does a lot more, producing a sequence of invariant for binary forms of even degree. Here, just want to look at one result, mentioned in the Wikipedia article.

One problem with reading old papers is that terminology has changed over the years. So for example, a binary form of degree n is homogeneous polynomial in two variables with degree n. A general binary form would be written as,

$$f(x,y) = a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{2}a_2x^{n-2}y^2 + \cdots + a_ny^n = \sum_{i=0}^{n} a_i\binom{n}{i}x^{n-i}y^i.$$

The coefficients a_i are arbitrary constants as usual. The inclusion of the binomial coefficients here was standard in the 19th century but has fallen out of use now. Here it is essential for the simplicity of the result. Finally here note that the ground field will be taken to be the complex numbers $\mathbb{C}$, again this is not usually specified in older work but is implicit since the fundamental theorem of algebra, that the complex numbers are complete, will be used.

2 The Theorem

A binary form of degree $2n$ can be split into a sum of powers of n linear forms if and only if its Catalecticant is zero.

Consider a binary form of even degree,

$$f(x,y) = \sum_{i=0}^{2n} a_i\binom{2n}{i}x^{2n-i}y^i,$$

the theorem give a condition for this to be equal to an expression of the form,

$$\phi(x, y) = \sum_{j=1}^{n} (p_j x + q_j y)^{2n}.$$

Notice that for a form of odd degree, $2n + 1$, there will be $2n + 2$ constants $a_0, \ldots, a_{2n+1}$. Hence when we equate this to a sum of powers of $n + 1$ linear factors, there will be the same number of constants to be determined. So we might expect that an odd degree binary form can always be decomposed into a sum of powers of $n + 1$ linear factors. And this is indeed the case as shown by Sylvester in an earlier paper. In the even case considered here there is one fewer constant to be determined than there are coefficients in the binary form. So in this case we would expect there to be a single condition on the coefficients which ensures the decomposition can be performed.

As an example consider the binary quartic,

$$f_e(x, y) = 2x^4 + 12x^3y + 30x^2y^2 + 36xy^3 + 17y^4.$$

This can be written as

$$f_e(x, y) = 2\binom{4}{0}x^4 + 3\binom{4}{1}x^3y + 5\binom{4}{2}x^2y^2 + 9\binom{4}{3}xy^3 + 17\binom{4}{4}y^4,$$

that is $a_0 = 2$, $a_1 = 3$, $a_2 = 5$, $a_3 = 9$ and $a_4 = 17$. The catalecticant is thus,

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{pmatrix} = 0.$$

Hence by the theorem, the fom can be written as the sum of two quartic of linear factors,

$$f_e(x, y) = (x + y)^4 + (x + 2y)^4.$$

(Of course the computations here were performed in the reverse order to the presentation above.)

3 Proof

Before expanding the sum of powers above let $q_j = \lambda_j p_j$ for $j = 1, \ldots, n$. So now we can write,

$$\phi(x, y) = \sum_{j=1}^{n} p_j^{2n} (x + \lambda_j y)^{2n}.$$

Expanding the powers of the linear factors gives,

$$\phi(x, y) = \sum_{i=0}^{2n} \left(\sum_{j=1}^{n} p_j^{2n} \lambda_j^i \right) \binom{2n}{i} x^{2n-i} y^i.$$

Comparing the coefficients between the above expansion and the definition of the form $f(x,y)$ gives $2n+1$ equations,

$$\begin{array}{cccccccc}
p_1^{2n} & + & p_2^{2n} & + & \cdots & + & p_n^{2n} & = & a_0, \\
p_1^{2n}\lambda_1 & + & p_2^{2n}\lambda_2 & + & \cdots & + & p_n^{2n}\lambda_n & = & a_1, \\
p_1^{2n}\lambda_1^2 & + & p_2^{2n}\lambda_2^2 & + & \cdots & + & p_n^{2n}\lambda_n^2 & = & a_2. \\
 & & & & & & & \vdots & \\
p_1^{2n}\lambda_1^{2n} & + & p_2^{2n}\lambda_2^{2n} & + & \cdots & + & p_n^{2n}\lambda_n^{2n} & = & a_{2n}.
\end{array}$$

Notice the cancellation of the binomial coefficients. The equations can be written in the matrix-vector form,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{2n} & \lambda_2^{2n} & \cdots & \lambda_n^{2n} \end{pmatrix} \begin{pmatrix} p_1^{2n} \\ p_2^{2n} \\ \vdots \\ p_n^{2n} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{pmatrix}. \qquad (1)$$

Now Take the first $n+1$ rows of this system,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n} & \lambda_2^{n} & \cdots & \lambda_n^{n} \end{pmatrix} \begin{pmatrix} p_1^{2n} \\ p_2^{2n} \\ \vdots \\ p_n^{2n} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n} \end{pmatrix}. \qquad (2)$$

The matrix on the left-hand side of this equation has order $(n+1)\times n$, hence there will be an $(n+1)$-vector, $(\Lambda_0,\, \Lambda_1, \ldots, \Lambda_n)$, which annihilates it:

$$(\Lambda_0,\, \Lambda_1, \ldots, \Lambda_n) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n} & \lambda_2^{n} & \cdots & \lambda_n^{n} \end{pmatrix} = 0.$$

In particular the elements Λ_i can be identified with the cofactors of the matrix,

$$\Lambda_0 = \det \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n} & \lambda_2^{n} & \cdots & \lambda_n^{n} \end{pmatrix}, \; \Lambda_1 = -\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n} & \lambda_2^{n} & \cdots & \lambda_n^{n} \end{pmatrix}, \ldots$$

up to,

$$\Lambda_n = (-1)^n \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}.$$

The last determinant here is the Vandermonde determinant and in fact the others are multiples of the Vandermonde determinant by a symmetric polynomial in the λ_i's. However, for our purposes, this is not important.

Multiplying equation (2) by the vector of cofactors gives a linear equation,

$$a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2 + \cdots + a_n\Lambda_n = 0.$$

Next we take another $n+1$ row from equation (1), this time starting from the second row,

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n+1} & \lambda_2^{n+1} & \cdots & \lambda_n^{n+1} \end{pmatrix} \begin{pmatrix} p_1^{2n} \\ p_2^{2n} \\ \vdots \\ p_n^{2n} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix}.$$

Notice that for any λ_i we have that,

$$\Lambda_0\lambda_i + \Lambda_1\lambda_i^2 + \cdots + \Lambda_n\lambda_i^{n+1} = \lambda_i(\Lambda_0 + \Lambda_1\lambda_i + \cdots + \Lambda_n\lambda_i^n) = 0,$$

and hence we get another linear homogeneous equation,

$$a_1\Lambda_0 + a_2\Lambda_1 + a_3\Lambda_2 + \cdots + a_{n+1}\Lambda_n = 0.$$

Clearly we can repeat this procedure until we get $n+1$ equations,

$$\begin{array}{ccccccccccc} a_0\Lambda_0 & + & a_1\Lambda_1 & + & a_2\Lambda_2 & + & \cdots & + & a_n\Lambda_n & = & 0, \\ a_1\Lambda_0 & + & a_2\Lambda_1 & + & a_3\Lambda_2 & + & \cdots & + & a_{n+1}\Lambda_n & = & 0, \\ & & & & & & & & & & \vdots \\ a_n\Lambda_0 & + & a_{n+1}\Lambda_1 & + & a_{n+2}\Lambda_2 & + & \cdots & + & a_{2n}\Lambda_n & = & 0. \end{array}$$

In matrix-vector form this is,

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix} \begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \vdots \\ \Lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3}$$

Now clearly the vanishing of the catalecticant,

$$\det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix} = 0,$$

gives a necessary condition for the decomposition to be possible, otherwise only trivial solution for the Λ_i's would be possible.

To show that this condition is also sufficient note that if it holds, then equation (3) has nontrivial solution. With the solutions for the Λ_i's we can find the λ_i's as the n solutions to the polynomial equation,

$$\Lambda_0 + \Lambda_1\lambda + \cdots + \Lambda_n\lambda^n = 0.$$

Finally the p_i^{2n}'s and hence the p_i's, can be found by linear algebra, that is from equation (1).

4 An Application

I found this problem in an old textbook, [2]. Show that the secant variety to the rational normal quartic curve is a cubic hypersurface. A hypersurface is an algebraic variety with one fewer dimensions than the projective space it lies in; a primal in older language.

The rational normal quartic curve can be thought of as a mapping from the projective line $\mathbb{P}^1$ to $\mathbb{P}^4$. In particular, if the line has homogeneous coordinates $(s : t)$ then the mapping is given by,

$$(s\,:\,t) \longrightarrow (s^4\,:\,s^3t\,:\,s^2t^2\,:\,st^3\,:\,t^4).$$

This can be seen as a parameterisation of the curve, with homogeneous parameters s and t. It is an example of a Veronese embedding, a general way to map one projective space into another of higher dimension.

If the $\mathbb{P}^4$ has homogeneous coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ then then the curve is given by the intersection of six quadric (degree 2) hypersurfaces. These can be expressed as,

$$\text{Rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} = 1.$$

That is the quadrics are given by the six degree 2 equations,

$$\begin{aligned} x_0x_2 - x_1^2 &= 0, \\ x_0x_3 - x_1x_2 &= 0, \\ x_0x_4 - x_1x_3 &= 0, \\ x_1x_3 - x_2^2 &= 0, \\ x_1x_4 - x_2x_3 &= 0, \\ x_2x_4 - x_3^2 &= 0. \end{aligned}$$

Now we can think of point in $\mathbb{P}^4$ as quartic binary forms, given a form:

$$a_0y^4 + 4a_1y^3x + 6a_2y^2x^2 + 4a_3yx^3 + a_4x^4,$$

we will associate the point,

$$(a_0\,:\,a_1\,:\,a_2\,:\,a_3\,:\,a_4) \in \mathbb{P}^4.$$

Note that multiplying the form by an overall non-zero constant doesn't change it, so these are points in a projective space.

Under this mapping forms which can be decomposed as the fourth power of a linear factor describe a rational normal quartic curve. To see this consider the fourth power of an arbitrary linear factor,

$$(px + qy)^4 = q^4y^4 + 4pq^3xy^3 + 6p^2q^2x^2y^2 + 4p^3qx^3y + p^4x^4,$$

where p and q are arbitrary. Such forms will be mapped to the points, $(q^4 : pq^3 : p^2q^2 : p^3q : p^4)$ in $\mathbb{P}^4$. That is they lie on a rational normal quartic curve.

A secant line to a curve is a line which meets the curve in two points. The closure of the set of these lines will also include tangent lines to the curve, where the two points, where the line meets the curve, coalesce. The set of points on all possible secant lines to an algebraic curve will form a three dimensional variety; two dimensions given by varying the points along the curve and another dimension as the point can move along the line. This means that for our quartic curve, its secant variety will be a hypersurface in $\mathbb{P}^4$ and hence it will be given by a single equation. Since the points on the curve correspond to forms which are decomposeable into single quartic factors, a point on a secant line to the curve will correspond to a linear combination of such quartics. So the condition for a quartic binary for to be decomposeable into the fourth powers of a pair of linear factor will be the same as the condition for the point in $\mathbb{P}^4$ to lie on the secant variety to the rational normal quartic curve. That is,

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0,$$

clearly a homogenous cubic in the coordinates of $\mathbb{P}^4$.

References

[1] Sylvester, J. J. (1852), "On the principles of the calculus of forms", *Cambridge and Dublin Mathematical Journal* vol. VII 52-97 and also 179–217. These can be found in the *The collected mathematical papers of J.J. Sylvester, Volume 1*, Cambridge University Press 1904.

[2] Semple, J.G. and Roth, L., *Introduction to algebraic geometry*, Oxford University Press 1985.