Graph homomorphisms II: some examples

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October 2006

Abstract

In this talk we will discuss some examples of graph homomorphisms. More precisely, the graph parameters which can be represented by counting the graph homomorphisms. The main reference is Section 2 in [2].

1 Introduction

In this note, we will study some of explicit examples about graph homomorphism, which provides a useful language and motivation to continue our study about [2]. The basic setting is as follows: $G = (V(G), E(G))$ is a simple graph unless stated otherwise, $\phi : G \to H$ is a homomorphism from $G$ to $H$ and $\text{hom}(G, H)$ is the number of homomorphisms from $G$ to $H$.

In fact, we should consider the homomorphisms both “from” $G$ and “to” $G$. The basic scheme in the paper [2] is:

$$F \to G \to H.$$

Given a (large, simple) graph $G$, we can study its local structure by counting of various “small” graphs $F$ into $G$; and its global structure by counting its homomorphisms into various small graph $H$. Roughly speaking, we can get some information about $G$ via “probing from the left with $F$”, which is related to property testing. On the other hand, we can “probing from the right with $H$, which is related to statistical physics. Informally, $H$ is also called the template (Model) of $G$. One useful observation is that any $\phi : G \to H$ gives a partition on $V(G)$ via the fibres of $\phi$. 
2 Weighted and unweighted

A weighted graph $H$ is a graph with a positive real weight $\alpha_H(i)$ associated with each node $i$ and a real weight $\beta_H(i, j)$ associated with each edge $ij$.

An edge with weight 0 will play the same role as no edge between those nodes, so we could assume that we only consider weighted complete graphs with loops at all nodes. An unweighted graph is a weighted graph where all the nodeweights and edgeweights are 1.

Let $G$ and $H$ be two weighted graphs. To every map $\phi : V(G) \rightarrow V(H)$, we assign the weight:

$$\text{hom}_\phi(G, H) = \prod_{uv \in E(G)} [\beta_{\phi(u)\phi(v)}(H)]^{\beta_{uv}(G)}$$

(1)

here ($0^0 = 1$). We then define

$$\text{hom}(G, H) = \sum_{\phi : V(G) \rightarrow V(H)} \alpha_\phi \text{hom}_\phi(G, H)$$

(2)

where

$$\alpha_\phi = \prod_{u \in V(G)} [\alpha_{\phi(u)}(H)]^{\alpha_u(G)}.$$  

(3)

We’ll use this definition most often in the case when $G$ is a simple unweighted graph, so that:

$$\alpha_\phi = \prod_{u \in V(G)} [\alpha_{\phi(u)}(H)].$$

and

$$\text{hom}_\phi(G, H) = \prod_{uv \in E(G)} [\beta_{\phi(u)\phi(v)}(H)]$$

3 Example

In this section we will present some examples about $\text{hom}(F, G)$ ($\text{hom}(G, H)$), where $F$ ($H$) is a single graph or a subfamily of finite graphs. The adjacent matrix of $G$ is denoted by $A(G) = (a_{ij})$, while $A^k = A(G^k) = (k a_{ij})$ is the adjacent matrix of $G^k$, the $k$ power graph of $G$. More precisely, $k a_{ij}$ is the number of walks from $i$ to $j$ and of length $k$. 
3.1 Left & unweighted

**Example 1.** Let $F$ be a point, then $\text{hom}(F, G) = |V(G)|$.
Let $F$ be $K_2$, then $\text{hom}(F, G) = 2|E(G)|$.
Let $F$ be $K_3$, then $\text{hom}(F, G) = 6 \times$ # of triangles. (note: here $G$ is assume to be simple.)

**Example 2.** Let $F$ be a $k$-path $P_k$, then $\text{hom}(P_k, G) = \#$ of walks with length $k - 1$. Here $(v_i, v_j, v_k)$ and $(v_k, v_j, v_i)$ should be treated as two different walks of length 3. In other words, $\text{hom}(P_k, G) = \sum_{i,j} (k-1a_{ij})$.

![Figure 1: the path on $k$ nodes](image1)

**Example 3.** Let $S_k$ be the star on $k$ nodes, then $\text{hom}(S_k, G) = \sum_{i=1}^{n} d_i^{k-1}$ where $d_i$ is the degree of $i \in V(G)$. Hence $\text{hom}(S_k, G)^{1/(k-1)}$ tends to the maximum degree of $G$ as $k \to \infty$.

**Proof.** Denote the vertices set of $S_k$ by $\{1, 2, \cdots, k\}$. Pick any $v \in V(G)$, and study the number of homomorphisms of $\phi : S_k \to G$ s.t $\phi(1) = v$. For each vertex of $S_k$ other than 1, there are $d_v$ different choices in $V(G)$ as its image where $d_v$ is the degree of $d_v$. Therefore we have totally $d_v^{k-1}$ such homomorphisms. On the other hand, the image of 1 can run through all vertices of $G$, therefore the number of homomorphisms between $S_k$ and $G$ is $\sum_{i=1}^{n} d_i^{k-1}$.

![Figure 2: the star on $k$ nodes](image2)
Example 4. Let $C_k$ be the cycles on $k$ nodes, then

$$\text{hom}(F, G) = \sum_{i=1}^{n} k a_{ii} = \sum_{i=1}^{n} \lambda_i^k$$

where $\lambda_i$ is the eigenvalue of $G$.

Proof. In this example, $G$ is not necessarily to be simple. Firstly we claim the number of loops in $G$ is equal to the sum of its eigenvalues. That is because this sum is the trace of $A_G$, the adjacent matrix of $G$, and we know the trace of $A_G$ counts the number of loops in $G$. Secondly we note that $\sum_{i=1}^{n} \lambda_i^k$ is the trace of $A_{G^k}$ where $V(G^k) = V(G)$ and $(i, j) \in E(G^k)$ iff there is exactly one path of length $k$ from $i$ to $j$ in the graph $G$. More precisely, $A_{G^k} = A(G) \times A(G) \cdots \times A(G)$. On the other hand, the number of loops in $G^k$ is the same as $\text{hom}(C_k, G)$, which complete the proof.

![Figure 3: the cycles on $k$ nodes](image)

**Example 5. (Random graphs)** Let $G(n, p)$ be a random graph with $n$ nodes and edgedensity $p$. Then for every simple graph $F$ with $k$ nodes,

$$E(\text{hom}(F, G)) = (1 + o(1))n^k p^{\binom{|E(F)|}{n}} \quad (n \to \infty).$$

Proof. Given any map $\phi$ from $F$ to $G$, denote the probability that $\phi$ is a homomorphism by $\rho$. Then we know $\rho = p^{\binom{|E(F)|}{n}}$ when $\phi$ is 1-1 and $p^{\binom{|E(F)|}{n}} \leq \rho \leq 1$ otherwise. On the other hand, the number of the 1-1 map from $F$ to $G$ is $n(n-1) \cdots (n-k+1)$ while the total number of the maps is $n^k$. Put all these together, we have:

$$E(\text{hom}(F, G)) = n(n-1) \cdots (n-k+1)p^{\binom{|E(F)|}{n}} + (n^k - n(n-1) \cdots (n-k+1))t$$
where \( p^{E(F)} | t \leq 1 \). Let \( \delta = t - p^{E(F)} \). Then \( 0 \leq \delta \leq 1 \) and we have:

\[
E(\text{hom}(F, G)) = n^k p^{E(F)} + (n^k - n(n - 1) \cdots (n - k + 1)) \delta
\]

\[
= (1 + o(1)) n^k p^{E(F)} \quad (n \to \infty)
\]

since

\[
\lim_{n \to \infty} \frac{(n^k - n(n - 1) \cdots (n - k + 1)) \delta}{n^k p^{E(F)}} = 0
\]

for fixed \( p, k, E(F) \) and \( 0 \leq \delta \leq 1 \). \( \square \)

### 3.2 Right & unweighted

**Example 6.** Let \( H \) be a point, then \( \text{hom}(G, H) \neq 0 \) iff \( G \) has no edge.

Let \( H \) be \( K_2 \), then \( \text{hom}(G, H) \neq 0 \) iff \( G \) is bipartite.

Let \( H \) be \( K_3 \), then \( \text{hom}(G, H) \neq 0 \) iff \( G \) is 3-colorable.

**Example 7.** (Independent Set) Let \( H \) be the graph on two nodes, with an edge connecting the two nodes and a loop at one of the nodes. Then \( \text{hom}(G, H) \) is the number of independent sets of nodes in \( G \).

**Proof.** The independent set is 1-1 corresponding to \( \phi^{-1}(2) \). More precisely, given any independent set \( A \), there is a unique \( \phi : G \to H \) such that \( A = \phi^{-1}(2) \). On the other hand, \( \phi^{-1}(2) \) is an independent set for any given homomorphism \( \phi : G \to H \). \( \square \)

Note: If \( H \) has only two nodes \( \{1, 2\} \), any \( \phi : G \to H \) is uniquely decided by any fibre of \( \phi \). (\( \phi^{-1}(1) \) or \( \phi^{-1}(2) \)) Or we can say such \( \phi \) given a partition on \( G \). In this sense there exists a 1-1 corresponding between the partitions and the homomorphisms where \( (V_1, V_2) \) and \( (V_2, V_1) \) are treated as two different partitions. In the above example, the part of partition corresponding to \( \phi^{-1}(2) \) is exactly the independent set, which is implied by the fact that point 2 is loopless.

**Example 8.** (Colorings) Let \( H \) be \( K_t \), then \( \text{hom}(G, H) = \# \) of the colorings of the graph \( G \) with \( t \) colors.

Note: We can consider the homomorphisms that mapped into a fixed graph \( H \) as generalized colorings, called the \( H \) colorings. There are also other generalizations, such as circular colorings and fractional colorings.
3.3 Weighted examples

Example 9. (Maximum cut) Let $H$ denote the looped complete graph on two nodes, weighted as follows: the non-loop edge has weight 8; all other edges and nodes have weight 1. Then for every simple graph $G$ with $n$ nodes,

$$\log_2 \text{hom}(G, H) - n \leq \text{MaxCut}(G) \leq \log_2 \text{hom}(G, H).$$

where $\text{MaxCut}(G)$ denotes the size of the maximum cut in $G$. So unless $G$ is very sparse, $\log_2 \text{hom}(G, H)$ is a good approximation of the maximum cut in $G$.

Proof. If $\{V_1, V_2\}$ is a partition of $V(G)$, the set $E(V_1, V_2)$ of all edges of $G$ crossing this partition is called a cut, whose size is denoted by $t_{\{V_1, V_2\}}$. From the discussion in Example 7, we known there is 1 – 1 corresponding between the partitions and the homomorphisms. On the other hand, the corresponding between partitions and cuts is many to one.
Choose a cut of maximal size, denote by $t_{\max}$, and a homomorphism $\phi_{\max}$ (not unique) corresponding to this cut. From the weight on $H$, we know:

$$\text{hom}(G, H) = \sum_{\phi: G \to H} \text{hom}_\phi(G, H) = \sum_{\phi: G \to H} 2^{t_{\phi}}$$

where $t_{\phi}$ is the cut size corresponding to the partition given by $\phi$.

Then we have

$$2^{t_{\max}} = \text{hom}_{\phi_{\max}}(G, H) \leq \text{hom}(G, H),$$

which means

$$\text{MaxCut}(G) \leq \log_2 \text{hom}(G, H).$$

On the other hand,

$$\text{hom}(G, H) \leq 2^n \text{hom}_{\phi_{\max}}(G, H) = 2^{n + t_{\max}}$$

as there are total $2^n$ homomorphisms from $G$ to $H$, which implies:

$$\log_2 \text{hom}(G, H) - n \leq \text{MaxCut}(G).$$

\qed

\textbf{Example 10. (Partition functions of the Ising model)} Let $G$ be any simple graph, and let $T > 0$, $h \geq 0$, and $J$ be three real parameters. Let $H$ be the looped complete graph on two nodes, denoted by $+$ and $-$, weighted as follows: $\alpha_+ = e^{h/T}$, $\alpha_- = e^{-h/T}$, $\beta_{++} = \beta_{--}$, and $\beta_{+-}/\beta_{-+} = e^{2J/T}$. Then $\text{hom}(G, H)$ is the partition function of the Ising model on the graph $G$ at temperature $T$ with coupling $J$ in external magnetic field $h$.

\textit{Proof.} The proof will be left as an exercise to the reader. (Hint: the configurations is 1-1 corresponding to the homomorphims via its fibres.) \qed

\subsection*{3.4 As a language}

- Chromatic Number:

$$\chi(G) = \min_k \{ k \mid \text{hom}(G, K_k) \neq 0 \}$$

- Clique Number:

$$\omega(G) = \max_n \{ k \mid \text{hom}(K_n, G) \neq 0 \}$$

- Odd girth:

$$og(G) = \min_l \{ 2l + 1 \mid \text{hom}(C_{2l+1}, G) \neq 0 \}$$
4 A nontrivial Example

4.1 Nowhere-zero flows

Let \( \Gamma \) be a finite abelian group and let \( S \) be a subset of \( \Gamma \) s.t \( S \) is closed under inversion.

For any graph \( G \), fix an orientation of the edges. An \( S \)-flow is an assignment of an element of \( S \) to each edge s.t for each node \( v \), the sum of elements assigned to edges entering \( v \) is the same as the sum of elements assigned to edges leaving \( v \).

Let \( f(G) \) be the number of \( S \)-flows. Then this number is independent of the choice of orientation.

Let \( S = \Gamma \setminus \{0\} \). Then such an \( S \)-flow is called nowhere-zero flow.

A Eulerian tour is an \( S \)-flow when \( \Gamma = \mathbb{Z}_2 \) and \( S = \mathbb{Z}_2 \setminus \{0\} \).

4.2 The representation of flows number

Let \( H \) be the complete directed graph (with all loops) on \( \hat{\Gamma} \). Let \( \alpha_\chi \triangleq \frac{a}{|\Gamma|} \) for each \( \chi \in \Gamma \), and let

\[
\beta_{\chi, \chi'} \triangleq \sum_{s \in S} \chi^{-1}(s)\chi'(s),
\]

for \( \chi, \chi' \in \hat{\Gamma} \). Then \( f(G) \) can be described as a homomorphism function [1].

Theorem 4.1. \( f(G) = \text{hom}(G, H) \).
The proof of this theorem is rather technical and will be put in the appendix. Instead, we will present two examples: the first one is the $H$ for the Eulerian characteristic function and the second is the $H$ for the Nowhere-zero 4 flows.

But we can sketch a proof for the special case for Eulerian tour, which in fact gives a new proof of the Eulerian theorem.

**Proof.**

• 1: If $G$ has Eulerian tour, then it can only have just one such tour.

• 2: If all vertices have even degree, then the size of any cut set is even.

• 3: If some vertices have odd degree, then the number of cut sets with even size equal to that of odd size.

  – The number of vertices having odd degree is even.
  – The parity of the cut size is totally decided by the parity of the vertices having odd degree.
  – Consider all possible partitions of $2k$ objects, the number of odd partition is equal to the even partition, which can be proven by induction.

• 4: Direct calculation gives the answer.
5 Some elementary properties

If $F$ is the disjoint union of two graphs $F_1$ and $F_2$, then

$$\text{hom}(F, G) = \text{hom}(F_1, G)\text{hom}(F_2, G).$$

If $F$ is connected and $G$ is the disjoint union of two graphs $G_1$ and $G_2$, then

$$\text{hom}(F, G) = \text{hom}(F, G_1) + \text{hom}(F, G_2).$$

Thus in a sense it’s enough to study homomorphisms between connected graphs.

For two simple graphs $G_1$, $G_2$, their categorical product $G_1 \times G_2$ is defined to be a graph with vertices set $V(G_1) \times V(G_2)$, in which $(i_1, j_1)$ is connected to $(i_2, j_2)$ iff $(i_1, i_2) \in E(G_1)$ and $(j_1, j_2) \in E(G_2)$. For this product, we have:

$$\text{hom}(F, G_1 \times G_2) = \text{hom}(F, G_1) \cdot \text{hom}(F, G_2).$$

6 Conclusion

Today we are focused on some examples of graph parameters which can be represented by counting the graph homomorphisms. A natural question would be: what are the parameters that are unable to have such representation? Surprisingly, the authors of [2] obtain some exact conditions to character such parameters, which would likely to be the topic of the next talk of this series. Or the reader can consult Section 3 in [2].

7 Appendix

7.1 A concrete example

In this subsection we will consider a concrete graph $G$, which is obtained by removing an edge from $K_4$.

For graph $G$, we have

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A^2(G) = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

On the other hand, we have

$$\det(\lambda I - A) = \lambda^4 - 5\lambda^2 - 4\lambda = \lambda(\lambda + 1)(\lambda^2 - \lambda - 4),$$
from which we can obtain the eigenvalues of $A(G)$:

$$
\lambda_1 = 0, \ \lambda_2 = -1, \ \lambda_3 = \frac{1}{2}(1 + \sqrt{17}), \ \lambda_4 = \frac{1}{2}(1 - \sqrt{17})
$$

Now we investigate which properties can we get by counting the homomorphisms from left and right. Firstly, from left:

- $\text{hom}(K_1, G) = |V(G)| = 4$.
- $\text{hom}(K_2, G) = 2|E(G)| = 10 = \sum_{i,j} a_{ij}$ where $K_2 = S_2 = P_2$.
- $\text{hom}(K_3, G) = 12 = 6 \times$ the number of triangles in $G$.
- $\text{hom}(P_3, G) = 26 = \sum_{i,j} (2a_{ij})$.
- $\text{hom}(S_3, G) = \sum_1^4 d_i^2 = 26$.
- $\text{hom}(C_2, G) = 10 = \sum_i (2a_{ii}) = \sum_i \lambda_i^2$.

For the right target, we use Independent Set to denote the graph $H$ we constructed as the target for the independent set and the same convenience for others. For the later use, we list the cut sizes and the partitions in the following table. (Note: this is unordered partition, in homomorphism we should use the ordered one).

<table>
<thead>
<tr>
<th>partition</th>
<th>{a}</th>
<th>{a,b}</th>
<th>{a,c}</th>
<th>{a,d}</th>
<th>{a,b,c}</th>
<th>{a,b,d}</th>
<th>{a,c,d}</th>
<th>{a,b,c,d}</th>
</tr>
</thead>
<tbody>
<tr>
<td>cut size</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

- $\text{hom}(G, K_2) = 0$, which means $G$ is not bipartite.
- $\text{hom}(G, K_3) = 6$, the number of 3 colors.
- $\text{hom}(G, \text{Independent Set}) = 6$, they are $\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{b, c\}\}$.
- $\text{hom}(G, \text{Maximum Cut}) = 2 \sum 2^{t_i} = 98$ while $\log_2 98 \approx 6$.
- $\text{hom}(G, \text{Eulerian}) = 2 \sum (\frac{1}{2})^4 (-1)^{t_i} = 0$. 

Figure 9: Welsh Graph
7.2 The Ising Model

Let $G = (V, E)$ be a finite graph, and call $\Omega = \{-1, +1\}^V$ the state space, with elements $\sigma = \{\sigma_x\}_{x \in V}$. The variable $\sigma_x \in \{-1, +1\}$ is called the spin at vertex $x$. This is a spin system.

There is an energy function defined on $\Omega$. For the Ising model this function is defined as:

$$H(\sigma) = -J \sum_{(x, y) \in E} \sigma_x \sigma_y - \sum_{x \in V} h \sigma_x.$$  

where $J$ is a real constant, the interaction strength, and $h \in \mathbb{R}$, decided by an external magnetic field.

Now we introduce the inverse temperature parameter $\beta \sim \frac{1}{T}$, and consider the following probability measure on $\Omega$:

$$\mu(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta}, \quad Z_\beta = \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)}. \quad (4)$$  

This $Z_\beta$ is called the partition function, and, as usually generating functions do, contains basically all information about the systems.

7.3 Group Characters

A character $\chi$ of $\Gamma$ is a homomorphism $\Gamma \rightarrow S^1$ where $S^1$ is the multiplicative group of complex numbers of modulus 1. The unit character $\chi_0$ is the character which assigns 1 to every element in $\Gamma$.

We list a few facts about characters:

- A function $\chi : \Gamma \rightarrow \mathbb{C}$ is a character iff it satisfies $\chi(a + b) = \chi(a) + \chi(b), a, b \in \Gamma$ since $\Gamma$ is finite.
- The set of all characters of $\Gamma$ form a group $\hat{\Gamma}$, called the dual group of $\Gamma$.
- $\hat{\Gamma} \cong \Gamma$.

Proposition 7.1.

$$\sum_{\chi \in \hat{\Gamma}} \chi(a) = \left\{ \begin{array}{ll} n & \text{if } a = 0 \\ 0 & \text{otherwise} \end{array} \right.$$  

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7.4 The proof of Theorem 4.1

Proof. Let \( n = |V(G)| \) and \( m = |\Gamma| \).

For any coloring \( \psi : E(G) \to S \) and node \( v \in V(G) \), let

\[
\partial_\psi(v) = \sum_{u \in V(G)} \psi(uv) - \sum_{u \in V(G)} \psi(vu).
\]

So \( \psi \) is an \( S \)-flow iff \( \partial_\psi(v) = 0 \).

Consider the expression

\[
A = \sum_{\psi : E(G) \to S} \prod_{v \in V(G)} \sum_{\chi \in \hat{\Gamma}} \chi(\partial_\psi(v)).
\] (5)

Form Proposition 7.1, the summation over \( \chi \) is 0 unless \( \partial_\psi(v) = 0 \), in which case it is \( m \). So the product over \( v \in V(G) \) is 0 unless \( \psi \) is an \( S \)-flow, in which case is it is \( m^n \). Therefore \( A \cdot m^{-n} \) counts \( S \)-flows.

On the other hand, we can expand the product over \( v \in V(G) \); this step looks like:

\[
\left( \chi_0(\partial_\psi(v_1)) + \chi_1(\partial_\psi(v_1)) + \cdots + \chi_{m-1}(\partial_\psi(v_1)) \right) \times \\
\left( \chi_0(\partial_\psi(v_2)) + \chi_1(\partial_\psi(v_2)) + \cdots + \chi_{m-1}(\partial_\psi(v_2)) \right) \times \\
\cdots \\
\bigg( \sum_{\phi}(\chi_{\phi(v_1)}(\partial_\psi(v_1)))\chi_{\phi(v_2)}(\partial_\psi(v_2))\cdots \chi_{\phi(v_n)}(\partial_\psi(v_n)) \bigg)
\]

Each term in the sum is corresponding to a choice of a character \( \chi_{\phi_v} \) for each \( v \). Denote \( \chi_{\phi_v} \) by \( \phi_v \) and so we get

\[
A = \sum_{\psi : E(G) \to S} \sum_{\phi : V(G) \to \hat{\Gamma}} \prod_{v \in V(G)} \phi_v(\partial_\psi(v)).
\]

Here (using that \( \phi_v \) is a character)

\[
\phi_v(\partial_\psi(v)) = \prod_{u \in V(G)} \phi_v(\psi(uv)) \prod_{u \in V(G)} \phi_u(\psi(uv))^{-1}
\]

So we get that

\[
A = \sum_{\psi : E(G) \to S} \sum_{\phi : V(G) \to \hat{\Gamma}} \prod_{uv \in E(G)} \phi_v(\psi(uv)) \phi_u(\psi(uv))^{-1}.
\]
Interchanging the summation, we have:

\[
A = \sum_{\phi: V(G) \to \hat{\Gamma}} \sum_{\psi: E(G) \to S} \prod_{uv \in E(G)} \phi_v(\psi(uv)) \phi_u(\psi(uv))^{-1}
= \sum_{\phi: V(G) \to \hat{\Gamma}} \prod_{uv \in E(G)} \sum_{s \in S} \phi_v(s) \phi_u(s)^{-1}
= \sum_{\phi: V(G) \to \hat{\Gamma}} \prod_{uv \in E(G)} \beta_{\phi(u), \phi(v)}
= \sum_{\phi: V(G) \to \hat{\Gamma}} m^n \text{hom}_\phi(G, H)
= m^n \text{hom}(G, H).
\]

where in the last two steps we use the following fact:

\[
\text{hom}(G, H) = \sum_{\phi: V(G) \to V(H)} (\alpha_\phi)(\text{hom}_\phi(G, H))
= \sum_{\phi: V(G) \to V(H)} (\prod_{u \in V(G)} [\alpha_{\phi(u)}(H)])(\prod_{uv \in E(G)} [\beta_{\phi(u), \phi(v)}(H)])
= \sum_{\phi: V(G) \to V(H)} \left(\frac{1}{m^n}\right)(\prod_{uv \in E(G)} [\beta_{\phi(u), \phi(v)}(H)]).
= \sum_{\phi: V(G) \to V(H)} \left(\frac{1}{m^n}\right)\text{hom}_\phi(G, H).
\]

Therefore, we have:

\[
f(G) = m^{-n} A = \text{hom}(G, H)
\]

which completes our proof.

\[\square\]

References
