Regular polytopes

Notes for talks given at LSBU, November & December 2014

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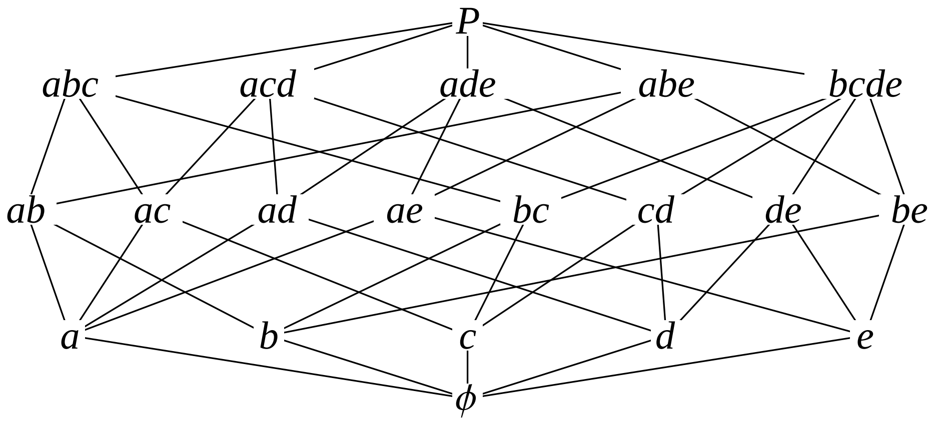
Flags A *flag* is a sequence $(f_{-1}, f_0, \ldots, f_n)$ of faces f_i of a polytope f_n, each incident with the next, with precisely one face from each dimension i, $i = -1, 0, \ldots, n$.

Since, however, the minimal face f_{-1} and the maximal face f_n must be in every flag, they are often omitted from the list of faces. The minimal face is often denoted by the empty set, $\emptyset$. In this face diagram of a square pyramid with apex a and base $bcde$ the flag on the right is $(\emptyset, e, be, bcde, P)$.



Vertex figure A vertex figure of an n-dimensional polytope is the $(n-1)$-dimensional polytope created when the vertex is sliced off by an $(n-1)$-hyperplane.

Truncation and rectification. For an n-polytope $\mathcal{P}$ and $t \le n-1$, truncation occurs when each vertex v is chopped off by an $(n-1)$-hyperplane that meets each edge, extended if necessary, incident with v at a point distance t from v. For special values of t we have *truncation*, $t = 1/3$; rectification, $t = 1/2$; bitruncation, $t = 2/3$; birectification, $t = 1$; and so on.

An m-rectification truncates m-faces to points. If an n-polytope is $(n-1)$-rectified, its $(n-1)$-faces are reduced to points and the polytope becomes its dual. For a 3-polytope, rectification reduces edges to points and birectification reduces faces to points.

For a 3-polytope, bitruncation is equivalent to truncation of its dual. Thus, for example,

$$\text{cube} \xrightarrow{t=\frac{1}{3}} \begin{array}{c}\text{truncated}\\ \text{cube}\end{array} \xrightarrow{t=\frac{1}{2}} \text{cuboctahedron} \xrightarrow{t=\frac{2}{3}} \begin{array}{c}\text{truncated}\\ \text{octahedron}\end{array} \xrightarrow{t=1} \text{octahedron}.$$

$$\text{octahedron} \xrightarrow{t=\frac{1}{3}} \begin{array}{c}\text{truncated}\\ \text{octahedron}\end{array} \xrightarrow{t=\frac{1}{2}} \text{cuboctahedron} \xrightarrow{t=\frac{2}{3}} \begin{array}{c}\text{truncated}\\ \text{cube}\end{array} \xrightarrow{t=1} \text{cube}.$$

The Schläfli symbol A regular convex p-gon as has Schläfli symbol $\{p\}$.

An n-polytope has Schläfli symbol $\{p_1, p_2, \ldots, p_{n-1}\}$ if its facets ($(n-1)$-faces) have Schläfli symbol $\{p_1, p_2, \ldots, p_{n-2}\}$ and the vertex figures have Schläfli symbol $\{p_2, p_3, \ldots, p_{n-1}\}$. Order is relevant.

Suppose $\mathcal{P}$ has Schläfli symbol $\{p_1, p_2, \ldots, p_n\}$. The dual of $\mathcal{P}$ has Schläfli symbol $\{p_n, p_{n-1}, \ldots, p_1\}$. A vertex figure of a facet of $\mathcal{P}$ is the same as a facet of a vertex figure of $\mathcal{P}$, $\{p_2, p_3, \ldots, p_{n-2}\}$.

Extensions of the Schläfli symbol A p-sided regular star polygon where the vertices are arranged in a circle and vertex i is joined to vertex $i+d$, $d < n/2$, $\gcd(d,p) = 1$, has Schläfli symbol $\{p/d\}$. Presumably p/d is not a number; so, for example, the pentagram must be written as $\{5/2\}$, not $\{2.5\}$.

A system of extensions to include Archimedean solids and the like consists of including the prefix t with subscripts. Suppose a polytope $\mathcal{P}$ has Schläfli symbol $\{p_1, p_2, \ldots, p_{n-1}\}$. Then

$t_0\mathcal{P} = \mathcal{P}$; $t_0\{4,3\} = \{4,3\}$ is a cube;

$t_1\mathcal{P}$ is a rectified $\mathcal{P}$; $t_1\{4,3\}$ is a cuboctahedron;

$t_2\mathcal{P}$ is a birectified $\mathcal{P}$; $t_2\{4,3\}$ is an octahedron (dual of the cube);

$t_3\mathcal{P}$ is a trirectified $\mathcal{P}$; $t_3\{4,3,3\}$ is a 16-cell (dual of the 4-cube);

$t_{0,1}\mathcal{P}$ is a truncated $\mathcal{P}$; $t_{0,1}\{4,3\}$ is a truncated cube;

$t_{0,2}\mathcal{P}$ is a rectified rectified $\mathcal{P}$; $t_{0,2}\{4,3\}$ is a rhombicuboctahedron;

$t_{1,2}\mathcal{P}$ is a bitruncated $\mathcal{P}$; $t_{1,2}\{4,3\}$ is a truncated octahedron;

$t_{0,1,2}\mathcal{P}$ is a truncated rectified $\mathcal{P}$; $t_{0,1,2}\{4,3\}$ is a truncated cuboctahedron.

However, this does not cover all Archimedean solids, so the notation is extended by further prefixes h, half, a, altered, as well as various modifications to the original symbol. For example, $ht_{0,1,2}\{4,3\}$ is a snub cube. See *Wiki* for details.

The groups S_n and A_n Label a tree on n vertices with the elements being permuted by S_n. Then by the result of Cara & Cameron, transpositions corresponding to the edges of the tree form an independent generating set for S_n. For $n = 1, 2$, $\mathrm{Aut}(S_n)$ and $\mathrm{Aut}(A_n)$ are trivial. For $n \geq 3$, $n \neq 6$, $\mathrm{Aut}(S_n) \cong S_n$ and all the automorphisms are inner. For $n = 6$, there is one outer automorphism, and $\mathrm{Aut}(S_6) \cong S_6 \rtimes C_2$. The outer automorphism can be defined by its action on generating transpositions:

$$(12) \mapsto (12)(34)(56),\ (13) \mapsto (13)(25)(46),\ (14) \mapsto (14)(26)(35),$$
$$(15) \mapsto (15)(24)(36),\ (16) \mapsto (16)(23)(45).$$

This works because $3 \cdot 5 = 15 = \binom{6}{2}$.

If $n \geq 4$, $n \neq 6$, $\mathrm{Aut}(A_n) \cong S_n$; $\mathrm{Aut}(A_3) = \mathrm{Out}(A_3) \cong C_2$ and $\mathrm{Aut}(A_6) \cong S_6 \rtimes C_2$. In each case the extra outer automorphism is conjugation by an odd permutation.

Partially ordered sets A *partially ordered set* is a set with a partial order $\leq$ that satisfies (i) $a \leq a$, (ii) $a \leq b$ & $b \leq a \Rightarrow a = b$ and (iii) $a \leq b$ & $b \leq c \Rightarrow a \leq c$.

A *join semi-lattice* is a partially ordered set where any two elements a, b have a supremum (or join or least upper bound), denoted by $a \vee b$. A bounded join semi-lattice has a unique maximum, usually denoted by 1.

A *meet semi-lattice* is a partially ordered set where any two elements a, b have an infimum (or meet or greatest lower bound), denoted by $a \wedge b$. A bounded meet semi-lattice has a unique minimum, usually denoted by 0.

A *lattice* is a partially ordered set where any two elements a, b have a supremum and an infimum. Thus it is both join semi-lattice and a meet semi-lattice. A bounded lattice has a unique maximum, 1, and a unique minimum, 0. The join and meet operations have the following properties.

$$a \vee b = b \vee a, \quad a \vee (b \vee c) = (a \vee b) \vee c, \quad a \vee (a \wedge b) = a, \quad a \vee a = a \vee 0 = a,$$

$$a \wedge b = b \wedge a, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad a \wedge (a \vee b) = a, \quad a \wedge a = a \wedge 1 = a.$$

A *Boolean algebra* or *Boolean lattice* is a complemented distributive lattice. The join and meet satisfy the distributivity law and there is a unary operation $\overline{a}$ (not a):

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee \overline{a} = 1, \quad a \wedge \overline{a} = 0.$$

A finite Boolean lattice is isomorphic to the lattice of subsets of a finite set X with $A \vee B = A \cup B$, $A \wedge B = A \cap B$ and $\overline{A} = X \setminus A$ for $A, B \subseteq X$. If $|X| = r$, we denote this lattice by $B(r)$.

Combinatorial polytopes An n-dimensional *polytope* $\mathcal{P}$ is a partially ordered set with the following properties. We call the elements *faces* and we say that two faces a and b are *incident* if $a \leq b$ or $b \leq a$.

(1) There exists a unique minimal element f_{-1}, and a unique maximal element f_n. Every maximal chain has length $n+2$ and contains both f_{-1} and f_n. These are the polytope's flags. If $(f_{-1}, f_0, \ldots, f_n)$, is a flag, then we say that f_i has *dimension* or *rank* i, $i = -1, 0, 1, \ldots, n$. A j*-face* is a face of dimension j. A *point* is a 0-face, an *edge* is a 1-face, a *polygon* is a 2-face and a *polyhedron* is a 3-face.

(3) If $a < b$ are faces of dimension i and $i+2$ respectively, then there exactly two faces of dimension $i+1$ incident with both a and b.

(4) Connectedness. If F and G are two flags, there is a sequence of flags starting at F and ending at G such that consecutive members intersect in all but one of their faces, and that $F \cap G$ is in each member.

Regular polytopes A polytope is regular if its automorphism group acts transitively on its flags. For an n-polytope, all the j-faces, $j = 0, 1, \ldots, n-1$ are regular polytopes of dimension less than n.

The regular polytopes realizable as convex objects in $\mathbb{E}^n$ are: (1) the closed interval;
(2) the regular m-gon, $m \geq 3$: $\{m\}$;
(3) the five Platonic solids: $\{3,3\}$, $\{4,3\}$, $\{3,4\}$, $\{5,3\}$, $\{3,5\}$;
(4) the six regular 4-polytopes: the 4-simplex $\{3,3,3\}$, the 4-cube $\{4,3,3\}$, the 16-cell or 4-cross-polytope $\{3,3,4\}$, the 24-cell $\{3,4,3\}$, the 120-cell $\{5,3,3\}$ and the 600-cell $\{3,3,5\}$;
(5) for $n \geq 5$, the n-simplex $\{3,3,\ldots,3\}$, the n-cube $\{4,3,3,\ldots,3\}$ and the n-cross-polytope $\{3,3,\ldots,3,4\}$.

A regular m-gon has $2m$ flags, the order of its automorphism group, D_m. A regular tetrahedron has 24 flags. Labelling its vertices $abcd$ and omitting the end points, the flags containing a are

$$(a, ab, abc), \quad (a, ab, abd), \quad (a, ac, abc), \quad (a, ac, acd), \quad (a, ad, abd), \quad (a, ad, acd).$$

Also one can identify the 48 flags in a cube. Labelling two opposite squares a, b, c, d and e, f, g, h such that ae, bf, cg, ch are edges, the six flags containing a are

$$(a, ab, abcd), \quad (a, ab, abfe), \quad (a, ad, abcd), \quad (a, ad, abhe), \quad (a, ae, abfe), \quad (a, ae, adhe).$$

Similarly we see that the octahedron has 48 flags, the dodecahedron 120 and icosahedron 120.

Coxeter groups

A *Coxeter group* is a group with presentation $\langle r_1, r_2, \ldots, r_n : (r_i r_j)^{m_{i,j}} = 1 \rangle$, where $m_{i,i} = 1$ and $m_{i,j} = m_{j,i} \geq 2$ for $i \neq j$. The condition $m_{i,j} = \infty$ means there is no condition $(r_i r_j)^m$.

The relation $m_{i,i} = 1$ means that $r_i^2 = 1$ for all i; the generators are involutions. If $m_{i,j} = 2$, the generators r_i and r_j commute. Observe that $(xy)^k$ and $(yx)^k$ are conjugates; $y(xy)^k y^{-1} = (yx)^k yy^{-1} = (yx)^k$.

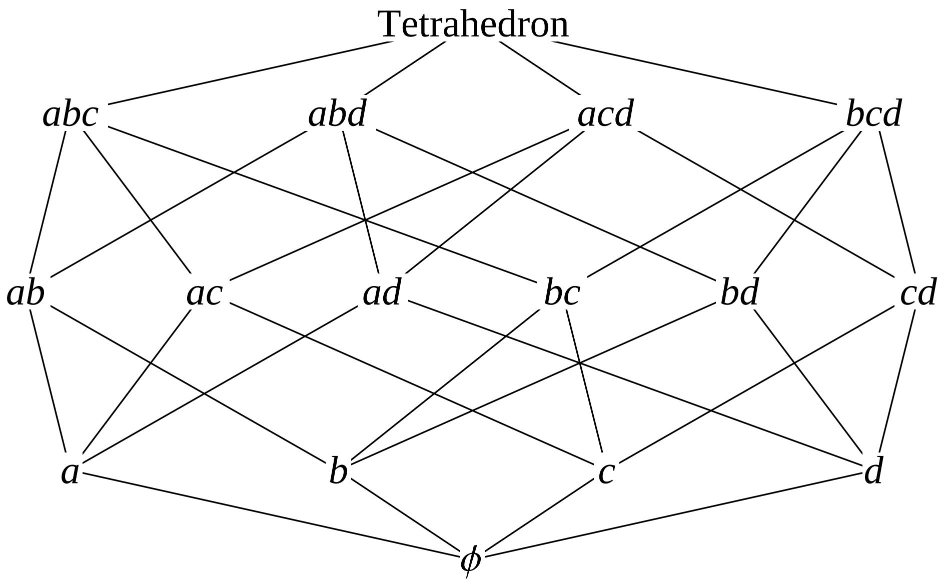
The *Coxeter matrix* is the $n \times n$ symmetric matrix with entries $m_{i,j}$. The Coxeter matrix can be encoded by a *Coxeter diagram*, an edge-labelled graph where the vertices are the generators

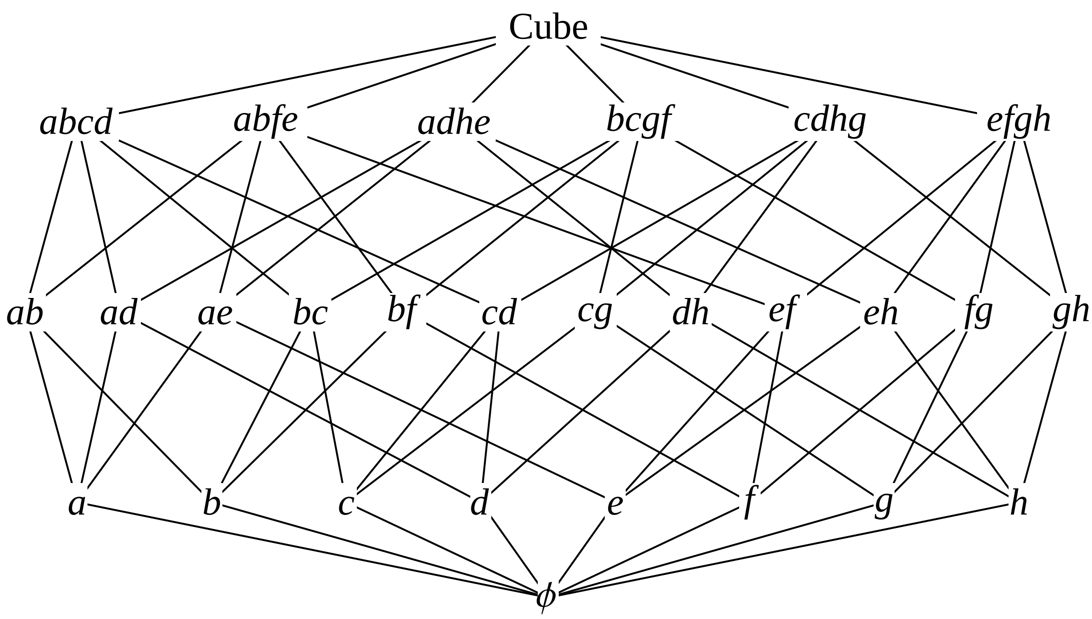
r_i, and $r_i \sim r_j$ iff $m_{i,j} \geq 3$; also the edge $r_i \sim r_j$ is labelled with the value of $m_{i,j}$ (usually omitted if 3). For example, the n-vertex path with edges labelled 3 (or unlabelled) gives the symmetric group S_{n+1}, the generators corresponding to $(1,2)$, $(2,3)$, $\ldots$, $(n-1,n)$, (n,∞).

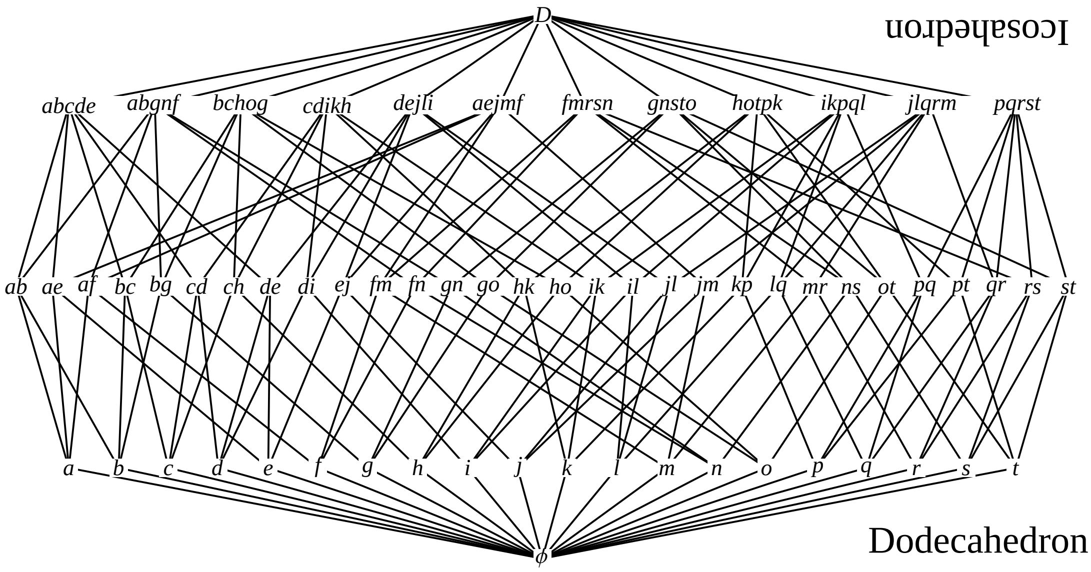
Polyhedra with symmetry group $I_h \cong A_5 \times C_2$ and rotation group $I \cong A_5$

Dodecahedron $\{5,3\}$, $v=20$, $e=30$, $f=12$; (note that $I_h \ncong S_5 \cong A_5 \rtimes C_2$)
Icosahedron $\{3,5\}$, $v=12$, $e=30$, $f=20$;
Great dodecahedron $\{5,5/2\}$, $v=12$, $e=30$, $f=12$;
Small stellated dodecahedron $\{5/2,5\}$, $v=12$, $e=30$, $f=12$;
Great icosahedron $\{3,5/2\}$, $v=12$, $e=30$, $f=20$;
Great stellated dodecahedron $\{5/2,3\}$, $v=20$, $e=30$, $f=12$.

Reference P. J. Cameron, Regular Polytopes, Slides for Old Codgers meeting, 2014.

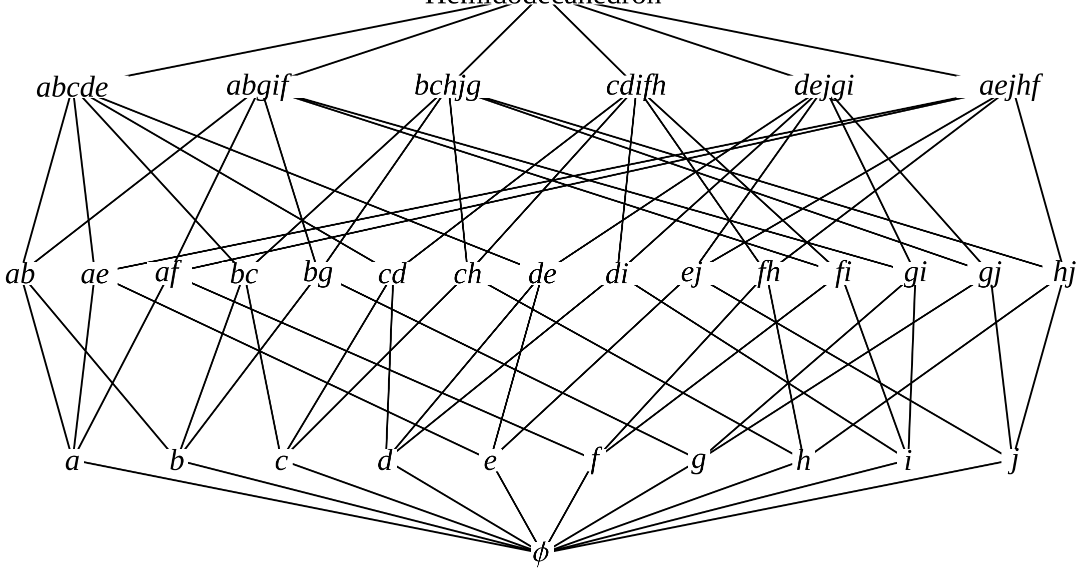




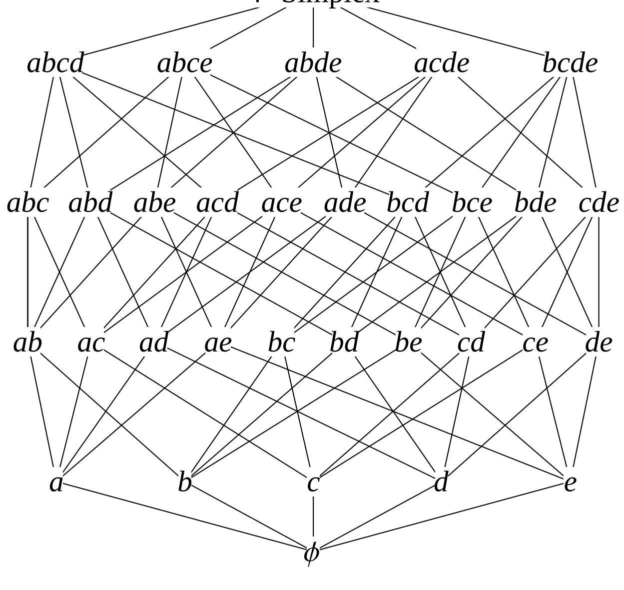


Dodecahedron

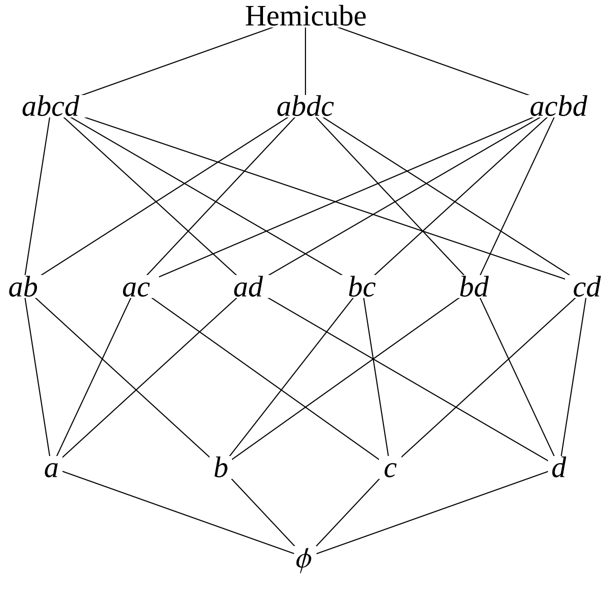
Hemidodecahedron

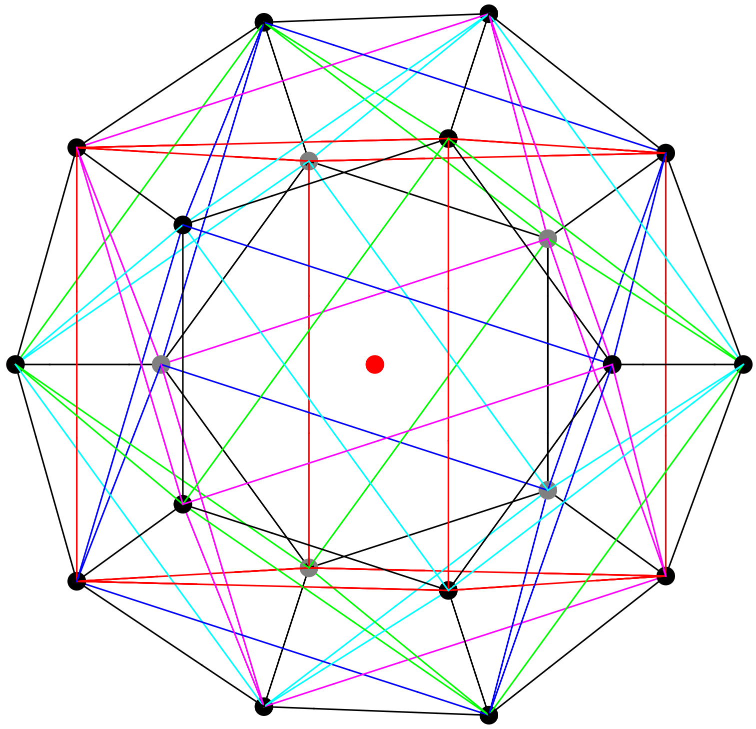


4−Simplex



Hemicube





Three views of a transparent regular dodecahedron. The 5 cubes formed by face diagonals are shown in red, green, blue, cyan and magenta. These are the objects on which the rotation group A_5 acts. Rotation by $2\pi/5$ induces a 5-cycle of the cubes; rotation by $2\pi/3$ fixed two cubes and 3-cycles the others; rotation by π fixes one cube and swaps the others in pairs.