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Which beta-shifts have a largest invariant measure?

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Abstract

For a given beta-shift, the lexicographic order induces a partial order (known as first-order stochastic dominance) on the collection of its shift-invariant probability measures. We characterize those beta-shifts for which this partial order has a largest element. These beta-shifts are all of finite type, and their lexicographically largest point is a periodic sequence of a particular kind: it is Sturmian (that is, its shift-orbit is combinatorially equivalent to a rotation) with weight-per-symbol either an integer, or equal to $p/(ap+1)$ for some $a, p \geqslant 1$, or equal to $A + p/(p+1)$ for some $p \geqslant 1$ and $A \geqslant 2$. In these cases, the largest invariant measure is precisely the unique one supported by the shift-orbit of the lexicographically largest point in the beta-shift.

1. *Introduction*

Rényi [**33**] considered the representation of real numbers with respect to an arbitrary base $\beta > 1$. These representations, the so-called *β-expansions*, are generated by orbits of the *beta-transformation* $T_\beta : x \mapsto \beta x \pmod 1$. The investigation of the ergodic properties of beta-transformations, and their relation to β-expansions, has been a fruitful area of research (see, for example, [**5**, **6**, **12**, **17**, **30**, **33**–**35**, **37**, **38**]). The present article is motivated by the following problem: Of the many T_β-invariant Borel probability measures on [0,1], which, if any, is *furthest to the right*?

To precisely formulate the notion of one probability measure being further to the right than another, we require the partial order of *first-order stochastic dominance* (see, for example, [**1**, Chapter 18.8]): if X is compact and totally ordered and μ and ν are Borel probability measures on X, then ν (*first-order stochastically*) *dominates* μ, written $\mu \prec \nu$, if $\mu(f) \leqslant \nu(f)$ for all increasing functions $f : X \to \mathbb{R}$. Equipping $X = [0, 1]$ with its usual order, a T_β-invariant probability measure would then be *furthest to the right* if it dominated all other T_β-invariant probability measures.

In fact, it is easily seen that the set of T_β-invariant probability measures never contains one that is furthest to the right: for example, when $\beta = 2$, the set of T_β-invariant probability measures has a supremum, namely the Dirac measure concentrated at the point 1, yet this measure is not T_β-invariant, since 1 is not a fixed point of T_β. This unsatisfactory solution, stemming from the non-compactness of the set of T_β-invariant probability measures, is easily remedied by working instead with the symbolic version[1] of T_β, its so-called *beta-shift*.

The (one-sided) beta-shift X_β is a subset of $\{0, \ldots, [\beta]\}^{\mathbb{N}}$, defined as the closure (with respect to the product topology) of the set of sequences arising as a β-expansion. The left shift map

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[1] The beta-shift is also the symbolic version of any map of the form $T(x) = F(x) \pmod 1$ for a differentiable real-valued function F with $F' > 1$ and $F(0) = 0$. Therefore the results of this paper have natural interpretations in terms of invariant probability measures for such maps T.

$\sigma : X_\beta \to X_\beta$, defined by $\sigma : (x_n)_{n=1}^\infty \mapsto (x_{n+1})_{n=1}^\infty$, is a continuous surjection, so the set $\mathcal{M}_{X_\beta}$ of σ-invariant Borel probability measures on X_β is weak-$*$ compact (see [**39**, Theorem 6.10]). After equipping X_β with the lexicographic order (this order is compatible with the one on $[0,1]$: if $(x_i)_{i=1}^\infty < (y_i)_{i=1}^\infty$ then $\sum_{i=1}^\infty x_i\beta^{-i} < \sum_{i=1}^\infty y_i\beta^{-i}$), the set $\mathcal{M}_{X_\beta}$ can be partially ordered by first-order stochastic dominance.

We may then ask: Which beta-shifts have a largest invariant measure? In other words, we wish to determine those $\beta > 1$ such that the partially ordered set $(\mathcal{M}_{X_\beta}, \prec)$ contains an element ν satisfying $\mu \prec \nu$ for all $\mu \in \mathcal{M}_{X_\beta}$. If $\beta > 1$ is an integer, then $X_\beta^+ := (\beta-1, \beta-1, \ldots)$ is the largest point in the corresponding beta-shift X_β, and also a fixed point for σ, so the following obviously holds.

THEOREM 1.1. *If $\beta > 1$ is an integer, then there exists a largest shift-invariant measure on the beta-shift X_β, namely the Dirac measure concentrated on the point $X_\beta^+ := (\beta-1, \beta-1, \ldots)$.*

Perhaps surprisingly, there also exist non-integer values of β such that X_β has a largest invariant measure. The purpose of this article is to determine the set of such values β.

For $1 < \beta < 2$, the characterization is as follows (note that in Theorems 1.2 and 1.3, w^k denotes k-fold concatenation of the word w; cf. Section 2.1.).

THEOREM 1.2. *For $1 < \beta < 2$, the following are equivalent:*

(i) *the beta-shift X_β has a largest shift-invariant measure;*

(ii) *the lexicographically largest element in X_β is the periodic sequence given by repeating the length-$(a\pi+1)$ word $(10^{a-1})^\pi 0$, for some integers $a, \pi \geqslant 1$;*

(iii) *β is the dominant root of the polynomial $\zeta^{a\pi+1} - \sum_{i=0}^{\pi} \zeta^{ia}$, for some integers $a, \pi \geqslant 1$ (by the* dominant *root we mean the largest real root).*

When the above conditions are satisfied, the largest shift-invariant measure on X_β is the unique one supported by the periodic shift-orbit generated by the largest element of X_β.

For $\beta > 2$, the existence of a largest invariant measure on X_β entails a more stringent restriction on the periodic word defining the largest point in X_β.

THEOREM 1.3. *For non-integer $\beta > 2$, the following are equivalent:*

(i) *the beta-shift X_β has a largest shift-invariant measure;*

(ii) *the lexicographically largest element in X_β is the periodic sequence given by repeating the length-$(\pi+1)$ word $B^\pi(B-1)$, for some integer $\pi \geqslant 1$, where B is the integer part of β;*

(iii) *β is the dominant root of the polynomial $\zeta^{\pi+1} - B\sum_{i=0}^{\pi} \zeta^i$, for some integers $\pi \geqslant 1$ and $B \geqslant 2$.*

When the above conditions are satisfied, the largest shift-invariant measure on X_β is the unique one supported by the periodic shift-orbit generated by the largest element of X_β.

REMARK 1.4. (i) If β satisfies the conditions of one of the Theorems 1.1–1.3, then X_β is, in particular, a subshift of *finite type* (see [**6**, pp. 136–138]).

(ii) The golden mean $\beta = (1+\sqrt{5})/2$ satisfies the condition of Theorem 1.2 for $a = \pi = 1$, so $X_{(1+\sqrt{5})/2}$ has a largest invariant measure, namely the one supported by its unique period-2 shift-orbit. More generally, if β is a so-called *multinacci* number (cf. [29]), that is, the dominant root of $\zeta^{\pi+1} - \sum_{i=0}^{\pi} \zeta^i$ for some $\pi \geqslant 1$, then Theorem 1.2 implies that X_β has a largest invariant measure.

(iii) If $\beta \approx 2.65897$, the dominant root of $\zeta^3 - 2\zeta^2 - \zeta - 2$, then X_β does not have a largest invariant measure, by Theorem 1.3, since the lexicographically largest element of X_β is the period-3 sequence $X_\beta^+ := 211211\ldots$, and 211 is not of the form $2^\pi 1$. As a direct verification of the absence of a largest invariant measure, note that $(\delta_{X_\beta^+} + \delta_{\sigma X_\beta^+} + \delta_{\sigma^2 X_\beta^+})/3$ gives strictly largest integral to the increasing function $\chi_{\{X_\beta^+\}}$ (the indicator function of $\{X_\beta^+\}$), but does not dominate the invariant probability measure supported by the period-2 orbit $\{0202\ldots, 2020\ldots\}$, since the latter measure gives larger integral to (for example) the indicator function for the set of points greater than or equal to $2020\ldots$.

(iv) The dominant root of $\zeta^4 - \zeta^3 - \zeta^2 - 1$ is $\beta \approx 1.75488$. The largest element of X_β is $X_\beta^+ := 11001100\ldots$, but 1100 is not of the form $(10^{a-1})^\pi\, 0$, so Theorem 1.2 implies that X_β does not have a largest invariant measure. This can be directly verified by showing (as in (iii) above) that the invariant measure supported by the orbit of X_β^+ is not dominated by any other invariant measure, but, on the other hand, does not dominate the invariant measure supported by the period-2 orbit $\{0101\ldots, 1010\ldots\}$.

(v) A comparison of (iii) and (iv) above gives some insight into the difference between Theorems 1.2 and 1.3: when $\beta > 2$, the luxury of an alphabet containing at least three symbols makes it easier to find invariant measures μ not dominated by the invariant measure generated by the largest element of X_β.

REMARK 1.5. Given a bounded measurable function $f : X_\beta \to \mathbb{R}$, a measure $\nu \in \mathcal{M}_{X_\beta}$ is called *f-maximizing* (see, for example, [3, 7, 21]) if $\mu(f) \leqslant \nu(f)$ for all $\mu \in \mathcal{M}_{X_\beta}$. Hence a largest invariant measure is one that is simultaneously f-maximizing for every increasing $f : X_\beta \to \mathbb{R}$.

For a comparable context see [22, 23], where certain invariant measures are shown to simultaneously optimize the integral of all *convex* functions (the associated partial order is known as *second-order stochastic dominance*, or alternatively as *majorization*). The optimizing measures in this case, so-called *Sturmian measures* (see Section 3; note that some authors prefer the term *balanced* rather than Sturmian), appear to be rather ubiquitous in certain problems of ergodic optimization (see also [3, 7, 8, 18, 19]).

In fact, in the present paper, the largest invariant measures of Theorems 1.1–1.3 are also Sturmian: these measures are supported by periodic orbits whose permutation (under the shift map) is a rotation. However, unlike in [3, 7, 8, 18, 19, 22, 23], not all rotation numbers play a role; rather, the relevant rotation numbers are those rationals of the form $\pi/(a\pi + 1)$ for integers $a, \pi \geqslant 1$.

Recall that an algebraic integer strictly larger than 1 is called a *Pisot number* if all of its algebraic conjugates have modulus strictly less than 1. Among those $\beta > 1$ such that X_β has a largest invariant measure, the following result characterizes those that are Pisot.

COROLLARY 1.6. *Let $\beta > 1$ be non-integer such that X_β has a largest invariant measure. The following are equivalent:*

(i) *β is a Pisot number;*

(ii) *β is the dominant root of $\zeta^{a\pi+1} - B\sum_{i=0}^{\pi} \zeta^{ia}$ for (π, a, B) equal to either $(1, 4, 1)$, $(1, 3, 1)$, or $(2, 3, 1)$, or $a = 1$ and $\pi, B \geqslant 1$, or $(a, B) = (2, 1)$ and $\pi \geqslant 1$;*

(iii) *X_β is of Sturmian type (see Section 3 for the definition), and the weight-per-symbol (see Section 2 for the definition) of its Sturmian measure is either $1/5$, $1/4$, or $2/7$, or $B - 1 + \pi/(\pi + 1)$ for integers $\pi, B \geqslant 1$, or $\pi/(2\pi + 1)$ for some integer $\pi \geqslant 1$.*

The organization of this article is as follows. Section 2 consists of preliminaries on symbolic dynamics, first-order stochastic dominance, and beta-shifts. Section 3 is devoted to Sturmian measures, and beta-shifts of Sturmian type, and includes the proof of Corollary 1.6. Theorems 1.1–1.3 are proved in Sections 4 and 5: beta-shifts without a largest invariant measure are dealt with in Theorem 4.5, while Theorem 5.4 pertains to beta-shifts that do have a largest invariant measure.

2. *Preliminaries*

2.1. *Symbolic dynamics*

NOTATION 2.1. Let $\mathbb{N}$ denote the set of strictly positive integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $m, n \in \mathbb{N}_0$, define $[m, n] := \{i \in \mathbb{N}_0 : m \leqslant i \leqslant n\}$.

DEFINITION 2.2 (Order, intervals, and topology). Define $F := \mathbb{N}_0^{\mathbb{N}}$, the set of all sequences $x = (x_n)_{n=1}^{\infty}$, where $x_n \in \mathbb{N}_0$ for all $n \in \mathbb{N}$. For $x, x' \in F$, we write $x < x'$ if there exists an $N \in \mathbb{N}$ with $x_N < x'_N$ and $x_n = x'_n$ for $1 \leqslant n < N$; we write $x \leqslant x'$ when $x = x'$ or $x < x'$. This *lexicographic order* $\leqslant$ is a total order on F. Throughout this article the set F, and certain of its subsets, will always be equipped with the order $\leqslant$; in particular, whenever we say that some sequence is less than, or greater than, another sequence, this will always be with respect to the lexicographic order.

For $X \subset F$ and $x, x' \in X$, define the *closed interval* $[x, x']_X := \{y \in X : x \leqslant y \leqslant x'\}$, and the *half-open interval* $(x, x']_X := \{y \in X : x < y \leqslant x'\}$. If the subset X is clear from the context then these intervals may be denoted, respectively, by $[x, x']$ and $(x, x']$.

The set $F = \mathbb{N}_0^{\mathbb{N}}$ will always be equipped with the product topology. The *shift map* $\sigma : F \to F$, defined by $(\sigma x)_n = x_{n+1}$ for $n \in \mathbb{N}$, is then continuous.

DEFINITION 2.3 (Words). A *word* is any element of the set $F \cup \bigcup_{n=0}^{\infty} \mathbb{N}_0^n$ (by convention, the *empty word* ε is the unique member of $\mathbb{N}_0^0$). By a *finite word* we mean an element of $\mathbb{N}_0^* = \bigcup_{n=0}^{\infty} \mathbb{N}_0^n$. It is usually convenient to suppress commas and braces, and write the finite word $w = (w_1, \ldots, w_n) \in \mathbb{N}_0^n$ as $w_1 \ldots w_n$. The *length* of w, denoted $|w|$, is n, while its *weight* is defined to be $w_1 + \ldots + w_n$. Its *weight-per-symbol* is defined to be $\frac{1}{n}(w_1 + \ldots + w_n)$.

Elements of F are called *infinite words* or *sequences*. For each $n \in \mathbb{N}$, define $\pi_n : F \to \mathbb{N}_0^n$ by $\pi_n(x) = (x_1, \ldots, x_n) = x_1 \ldots x_n$. A length-$m$ word w is said to be a *factor* of the infinite word $x = (x_i)_{i=1}^{\infty}$ if $w = x_j x_{j+1} \ldots x_{j+m-1}$ for some $j \in \mathbb{N}$.

A sequence $x \in F$ is called *periodic* if there exists a $p \in \mathbb{N}$ such that $x_n = x_{n+p}$ for all $n \in \mathbb{N}$; in this case the smallest such p is called the *period* of x, and $\pi_p(x)$ is called the corresponding *periodic word*. Conversely, given a finite word $w = w_1 \ldots w_p$, the *periodic sequence determined by* w, denoted by $\overline{w}$, is the sequence $x = (x_n)_{n=1}^{\infty}$ such that $x_n = w_{n \,(\mathrm{mod}\ p)}$ for all $n \in \mathbb{N}$. The *periodic orbit* determined by w is the orbit $\{\sigma^i(\overline{w}) : 0 \leqslant i \leqslant p-1\}$.

DEFINITION 2.4 (Concatenation). For a finite word w and any word x, their *concatenation* wx is the word defined by $(wx)_i = w_i$ for $i \in [1, |w|]$, and $(wx)_i = x_{i-|w|}$ for $i > |w|$. Under concatenation, the set $\mathbb{N}_0^*$ is the free monoid on $\mathbb{N}_0$, with ε as its identity element; $\mathbb{N}_0^* \setminus \{\varepsilon\}$ is the free semi-group on $\mathbb{N}_0$.

NOTATION 2.5. If $w \in \mathbb{N}_0^*$, $n \in \mathbb{N}$, and $G \subset F$, then define $w^n := \pi_{n|w|}(\overline{w})$ and $wG := \{wx : x \in G\}$.

DEFINITION 2.6 (Subshifts). For any $k \in \mathbb{N}$, define $F_k := \{0, \ldots, k-1\}^{\mathbb{N}}$. Clearly, $(F_k)_{k=1}^{\infty}$ is an increasing sequence of sets, with $\bigcup_{k=1}^{\infty} F_k = l^{\infty}(\mathbb{N}_0)$, the set of all bounded sequences with entries in $\mathbb{N}_0$. When equipped with the product topology, each F_k is compact. Clearly, the smallest element in $(F_k, \leqslant)$ is $\overline{0} = (0, 0, 0, \ldots)$, and the largest element is $\overline{k-1} = (k-1, k-1, k-1, \ldots)$.

Any non-empty closed subset $X \subset F_k$ satisfying $\sigma(X) = X$ is called a *subshift* (of F_k). For any subshift X the restricted shift map $\sigma|_X$ is a continuous endomorphism of X.

NOTATION 2.7 (Largest element). If X is any subshift then $(X, \leqslant)$ has a largest element, denoted by X^+. If X^+ is periodic, then let ζ_X denote its periodic word. Define $m_X := \pi_1(X^+) \in \mathbb{N}_0$, the largest integer appearing as an entry in elements of X. Any interval in X whose right endpoint is X^+ is called an *upper interval* in X.

NOTATION 2.8 (Cylinder set and invariant measures). For a subshift X and $w \in \mathbb{N}_0^*$, define $\langle w \rangle_X := \{x \in X : \pi_{|w|}(x) = w\}$, the associated *cylinder set* (and, in particular, a closed interval in X).

Let $\mathcal{M}_X$ denote the collection of σ-invariant Borel probability measures on X.

2.2. *Dominance*

DEFINITION 2.9. For X a subshift and $\mu, \nu \in \mathcal{M}_X$, we say that μ is *dominated* by ν (or ν *dominates* μ), and write $\mu \prec \nu$, if $\mu(f) \leqslant \nu(f)$ for every increasing function $f : X \to \mathbb{R}$ (a function $f : X \to \mathbb{R}$ is said to be increasing if $f(x) \leqslant f(x')$ whenever $x \leqslant x'$).

It is easily shown that $\prec$ defines a partial order on $\mathcal{M}_X$, usually called *first-order stochastic dominance*. We use abbreviated terminology here because higher-order stochastic dominance will not be considered. See [**22**, **23**] for the usage of *second-order* stochastic dominance (also variously known as *majorization*, *dilation*, or *balayage*) in an ergodic theory setting.

Simple approximation arguments (cf., for example, [**25**] or [**27**, Proposition 17.A.1]) yield the following useful reformulations of dominance.

LEMMA 2.10. *For X a subshift and $\mu, \nu \in \mathcal{M}_X$, the following are equivalent:*

(i) $\mu \prec \nu$;
(ii) $\mu(x, X^+]_X \leqslant \nu(x, X^+]_X$ *for all* $x \in X$;
(iii) $\mu[x, X^+]_X \leqslant \nu[x, X^+]_X$ *for all* $x \in X$.

DEFINITION 2.11. Let X be a subshift. A measure $\mu \in \mathcal{M}_X$ is said to be the largest (respectively, *smallest*) *invariant measure* on X if it is the largest (respectively, smallest) element in $(\mathcal{M}_X, \prec)$, that is, if $\nu \prec \mu$ (respectively, $\mu \prec \nu$) for all $\nu \in \mathcal{M}_X$.

REMARK 2.12. (i) A subshift X need not have either a largest or smallest invariant measure. Each F_k, however, has both largest and smallest invariant measures, namely the Dirac measures concentrated on the fixed points $\overline{k-1} = (k-1, k-1, k-1, \ldots)$ and $\overline{0} = (0, 0, 0, \ldots)$.

(ii) For a Borel subset $Y \subset X$, a corresponding *measure of maximal hitting frequency* is defined (cf. [**20**]) to be any measure $\nu \in \mathcal{M}_X$ satisfying $\nu(Y) = \sup_{\mu \in \mathcal{M}_X} \mu(Y)$. Lemma 2.10 implies that $\nu \in \mathcal{M}_X$ is a largest invariant measure on X if and only if it is the measure of maximum hitting frequency for each upper interval in X.

2.3. *Beta-shifts*

DEFINITION 2.13. For $x \in F$, the corresponding (one-sided) *beta-shift* is defined to be the largest closed, shift-invariant subset of $[\overline{0}, x]_F$, that is, the set

$$\bigcap_{n=0}^{\infty} \sigma^{-n} \left[\overline{0}, x\right]_F . \tag{1}$$

Denote the collection of all beta-shifts by $\mathfrak{B}$.

REMARK 2.14. (i) Distinct choices of x in (1) may yield the same beta-shift X; clearly, the smallest possible choice is $x = X^+$.

(ii) Clearly, if $X, Y \in \mathfrak{B}$ then $X \subsetneq Y$ if and only if $X^+ < Y^+$.

(iii) Every beta-shift X is contained in F_{m_X}, and hence in F_k for every $k > m_X$.

(iv) According to Definition 2.13, the singleton $\{\overline{0}\}$ is a beta-shift. All other beta-shifts are Cantor sets.

(v) By a result of Parry [30], Definition 2.13 is essentially (the only difference is our convention of including $\{\overline{0}\}$ as a beta-shift) equivalent to the definition of a beta-shift mentioned in Section 1, in terms of β-expansions (see, for example, [6, 30, 33]). Fixing a real number $\beta > 1$, for each $t \in [0, 1)$ define $d_n^\beta(t) := [\beta T_\beta^{n-1}(t)]$ for $n \geqslant 1$, where $T_\beta(t) := \beta t \pmod 1$. The *base-$\beta$ greedy expansion* of t is the sequence $d^\beta(t) := (d_n^\beta(t))_{n=1}^\infty \subset F_{[\beta]}$. If X_β denotes the closure in $F_{[\beta]}$ of $d^\beta([0, 1))$, then there is a unique $X \in \mathfrak{B}$ and $\beta > 1$ such that $1 = \sum_{i=1}^\infty (X^+)_i \beta^{-i}$, and hence $X_\beta = X$. Reciprocally, for every $X \in \mathfrak{B} \setminus \{\overline{0}\}$ there exists a $\beta > 1$ with $X_\beta = X$.

(vi) Every $X \in \mathfrak{B}$ contains the fixed point $\overline{0}$, and $\delta_{\overline{0}}$ is clearly the smallest element in $(\mathcal{M}_X, \prec)$. It is less obvious whether or not $(\mathcal{M}_X, \prec)$ has a *largest* element.

The following is immediate from Definition 2.13.

LEMMA 2.15. *Let $X \in \mathfrak{B}$ and let $x \in F$. If $\sigma^n(x) \leqslant x \leqslant X^+$ for all $n \geqslant 0$, then the orbit $\{\sigma^n(x) : n \geqslant 0\}$ is contained in X.*

NOTATION 2.16. For $X = \{\overline{0}\}$, define $X^- := \overline{0}$. For $X \in \mathfrak{B} \setminus \{\overline{0}\}$, define $X^- \in X$ by

$$(X^-)_1 := m_X - 1 \quad \text{and} \quad (X^-)_n := (X^+)_n \ \text{ for } n \geqslant 2.$$

REMARK 2.17. For $X \in \mathfrak{B}$, each $x \in X$ has precisely one preimage[2] in $(X^-, X^+]_X$: if $x \in [\overline{0}, \sigma(X^+)]_X$ then $m_X x \in X$, while if $x \in (\sigma(X^+), X^+]_X$ then $(m_X - 1)x \in X$.

Each $x \in [\overline{0}, \sigma(X^+)]_X$ has precisely $m_X + 1$ preimages, while each $x \in (\sigma(X^+), X^+]_X$ has precisely m_X preimages.

3. *Sturmian measures and beta-shifts of Sturmian type*

LEMMA 3.1. *For $X \in \mathfrak{B}$, precisely one measure in $\mathcal{M}_X$ has its support contained in the interval $[X^-, X^+]_X$.*

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[2] A *preimage* always means a preimage under $\sigma|_X$; so to say that x has l preimages means that $(\sigma|_X)^{-1}(x)$ has cardinality l.

Proof. Existence is straightforward, since $[X^-, X^+]_X$ is a closed set whose shift image is X, while the proof of uniqueness is almost identical to that of Bousch and Mairesse [8] for the case $X = X_2 = F_2$ (cf. also [9]). □

DEFINITION 3.2. A shift-invariant Borel probability measure on F is called *Sturmian* if its support lies in $[X^-, X^+]_X$ for some $X \in \mathfrak{B}$. If we wish to emphasize the beta-shift X, we refer to the Sturmian measure for X, and denote it by S_X.

REMARK 3.3. The map $X \mapsto S_X$ on $\mathfrak{B}$ is not one-to-one: any Sturmian measure supported on a periodic orbit (with the exception of the fixed point $000\ldots$) is the Sturmian measure for (uncountably) many beta-shifts. For example, the Dirac measure concentrated on $111\ldots$ is the Sturmian measure for X_β for all $2 \leqslant \beta \leqslant (3+\sqrt{5})/2$.

DEFINITION 3.4. For a Sturmian measure S_X, call $\sum_{n\geqslant 1} nS_X(\langle n\rangle_X)$ its *weight-per-symbol*, and note that this equals

$$(m_X - 1)S_X(\langle m_X - 1\rangle_X) + m_X S_X(\langle m_X\rangle_X) = S_X(\langle m_X\rangle_X) + m_X - 1, \tag{2}$$

because $[X^-, X^+]_X \subset \langle m_X - 1\rangle_X \cup \langle m_X\rangle_X$.

The following standard properties of Sturmian measures will be required.

PROPOSITION 3.5. (i) *For every $\varrho \in [0,\infty)$ there exists a unique Sturmian measure whose weight-per-symbol equals ϱ; we denote this measure by S_ϱ.*

(ii) *The measure S_0 is the Dirac measure concentrated on the fixed point $000\ldots$.*

(iii) *For $\varrho \in (0,1)$, if $R_\varrho(t) := t + \varrho \pmod 1$, and*

$$x_n^{(\varrho)}(t) := \begin{cases} 0 & \text{if } R_\varrho^{n-1}(t) \in [0, 1-\varrho), \\ 1 & \text{otherwise}, \end{cases}$$

then S_ϱ is the push-forward of Lebesgue measure on $[0,1)$ under the map $t \mapsto (x_n^{(\varrho)}(t))_{n=1}^\infty$.

(iv) *We have $S_\varrho = S_{\varrho-[\varrho]} \circ i_{[\varrho]}$ for $\varrho \in [1,\infty)$, where $i_{[\varrho]}$ is defined by $i_{[\varrho]} : (x_n)_{n=1}^\infty \mapsto (x_n - [\varrho])_{n=1}^\infty$.*

(v) *If ϱ is irrational, then the support of S_ϱ is a uniquely ergodic Cantor set. If ϱ is rational then S_ϱ is supported on a single periodic orbit; writing $\varrho = A + \pi/q$, with $A \in \mathbb{N}_0$, $q \in \mathbb{N}$, $\pi \in [0, q-1]$, and $\gcd(\pi, q) = 1$, if the points in this orbit are $s_1 < \ldots < s_q$, then the shift σ acts as a cyclic permutation:*

$$\sigma(s_i) = \begin{cases} s_{i+\pi} & \text{if } i \in [0, q-\pi], \\ s_{i+\pi-q} & \text{if } i \in [q-\pi+1, q], \end{cases}$$

that is, it is combinatorially equivalent to the action of $R_{\pi/q}(t) := t + \pi/q \pmod 1$ on any of its periodic orbits.

(vi) *Any point in the support of a Sturmian measure is* balanced*; that is, for each $a \in \mathbb{N}_0$, and all pairs of factors u and v of x of equal length, we have $||u|_a - |v|_a| \leqslant 1$, where $|u|_a$ and $|v|_a$ denote the number of occurrences of a in u and v, respectively. If a periodic sequence is balanced, and contained in $\{A, A+1\}^{\mathbb{N}}$ for some $A \in \mathbb{N}_0$, then it is an atom of a (unique) Sturmian measure.*

Proof. For (ii), observe that the Dirac measure $\delta_{\overline{0}}$ is the Sturmian measure for the beta-shift $\{\overline{0}\}$. For (iii) see, for example, [3, 7–9, 22, 23], though the connection with rotations goes back to [28]. For (iv), if $\tau \in [0,1)$ and $\mathrm{supp}(S_\tau) \subset [X^-, X^+]_X$ for some $X \in \mathfrak{B}$, then, for any $A \in \mathbb{N}$, the support of $S_\tau \circ i_A$ lies in $[Y^-, Y^+]_Y$, where $Y \in \mathfrak{B}$ is defined by $(Y^+)_n = (X^+)_n + A$ for $n \in \mathbb{N}$. Property (i) then follows from the existence and uniqueness of S_ϱ for $\varrho \in [0,1)$ (see [8, 9]), and (v) follows from (iii) and (iv) (cf. also [3, 7–9, 22, 23]). The balanced property (vi) is well known; see, for example, [28] or [32, Chapter 6]. □

REMARK 3.6. Sturmian measures appear explicitly in [3, 7, 8, 18, 19, 22, 23] in the case $\beta = 2$, while the terminology *Sturmian* goes back to the description by Morse and Hedlund [28] of the points in the support of Sturmian measures, that is, so-called *Sturmian sequences*. (As indicated by Proposition 3.5(vi), some authors prefer the term *balanced*, reserving the terminology Sturmian for the case of irrational ϱ (see, for example, [2, 4, 32])). For general $\beta > 1$, Sturmian sequences are considered in [10, 24, 26].

NOTATION 3.7. For every $\varrho \in [0,\infty)$, there is clearly a unique $X(\varrho) \in \mathfrak{B}$ such that $X(\varrho)^+ = \max \mathrm{supp}(S_\varrho)$ (note that, with this notation, $S_{X(\varrho)} = S_\varrho$). Any such beta-shift $X(\varrho)$ is said to be of *Sturmian type*.

If $\varrho \in [0,\infty)$ is rational, then we use $\zeta(\varrho)$ as shorthand for $\zeta_{X(\varrho)}$, the periodic word for $X(\varrho)^+$.

REMARK 3.8. If X is a beta-shift of Sturmian type, and S_X is periodic, then X^- is strictly pre-periodic, so the support of S_X is contained in the half-open interval $(X^-, X^+]_X$.

The largest point in the support of a Sturmian measure can be generated by a concatenation procedure analogous to the Farey construction of rational numbers (see, for example, [15, Chapter III]). For $n \geqslant 1$, let $\mathfrak{F}_n$ denote the *order-n Farey sequence*, that is, the increasing finite sequence consisting of rationals in $[0,1]$ whose denominator is at most n, where 0 and 1 are included as $\frac{0}{1}$ and $\frac{1}{1}$, respectively. For example, $\mathfrak{F}_5$ is $\frac{0}{0}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$. Two rationals in $[0,1]$ are said to be *Farey-consecutive* if they appear as successive terms in some $\mathfrak{F}_n$.

LEMMA 3.9. *If $p_1/q_1 < p_2/q_2$ (with $p_1, p_2, q_1, q_2 \in \mathbb{N}$ and $\gcd(p_1,q_1) = 1 = \gcd(p_2,q_2)$) are Farey-consecutive, then*

$$\zeta\left(\frac{p_1+p_2}{q_1+q_2}\right) = \zeta\left(\frac{p_2}{q_2}\right)\zeta\left(\frac{p_1}{q_1}\right).$$

If $\varrho \in [0,1]$ is irrational and $\{\varrho_i\}$ is any sequence of rationals converging to ϱ, then $X(\varrho)^+$ is the limit, in F_2, of $X(\varrho_i)^+$.

If $\varrho \in (1,\infty)$ then $X(\varrho)^+$ is given by $(X(\varrho)^+)_i = (X(\varrho - [\varrho])^+)_i + [\varrho]$ for all $i \geqslant 1$.

Proof. The proof is easily adapted from the one in [31] (which treats the *smallest* point in the support of the Sturmian measure); see also, for example, [19]. □

EXAMPLE 3.10. *If* $A \in \mathbb{N}_0$ and $B = A+1$, then $\zeta(A+1/2)$, $\zeta(A+2/3)$, $\zeta(A+3/5)$, $\zeta(A+5/8)$, and $\zeta(A+8/13)$ are, respectively,

$$BA\,,\ BBA\,,\ BBABA\,,\ BBABBABA\,,\ BBABBABABBABA\,,$$

while

$$X(A+(1+\sqrt{5})/2)^+ = BBABBABABBABBABABBABABBABBABABBABA\ldots\,.$$

NOTATION 3.11. Let $\beta(\varrho)$ denote the unique $\beta > 1$ that satisfies $X_\beta = X(\varrho)$.

The functions mapping ϱ to $X(\varrho)^+$, $X(\varrho)$, and $\beta(\varrho)$ are all increasing.

LEMMA 3.12. *For $\varrho, \varrho' \in [0, \infty)$, we have*

$$\varrho < \varrho' \iff X(\varrho)^+ < X(\varrho')^+ \iff X(\varrho) \subsetneq X(\varrho') \iff \beta(\varrho) < \beta(\varrho').$$

Proof. For the first equivalence, recall (cf. Notation 3.7) that the beta-shift $X(\tau)$ is defined by the equality $X(\tau)^+ = \max \operatorname{supp}(S_\tau)$, and it is well known that $\max \operatorname{supp}(S_\varrho) < \max \operatorname{supp}(S_{\varrho'})$ if and only if $\varrho < \varrho'$ (see [8] or [9]). The second equivalence is immediate from the definition of a beta-shift (cf. Remark 2.14(ii)), while the third equivalence follows from the well-known fact (see, for example, [6]) that $\beta < \beta'$ if and only if $X_\beta \subsetneq X_{\beta'}$. □

We are now able to prove Corollary 1.6, which, for convenience, is re-stated here.

COROLLARY 1.6. *Let $\beta > 1$ be non-integer such that X_β has a largest invariant measure. The following are equivalent:*

(i) *β is a Pisot number;*
(ii) *β is the dominant root of $\zeta^{a\pi+1} - B\sum_{i=0}^{\pi} \zeta^{ia}$ for (π, a, B) equal to either $(1, 4, 1)$, $(1, 3, 1)$, or $(2, 3, 1)$, or $a = 1$ and $\pi, B \geqslant 1$, or $(a, B) = (2, 1)$ and $\pi \geqslant 1$;*
(iii) *X_β is of Sturmian type, and the weight-per-symbol of its Sturmian measure is either $1/5$, $1/4$, or $2/7$, or $B - 1 + \pi/(\pi + 1)$ for integers $\pi, B \geqslant 1$, or $\pi/(2\pi + 1)$ for some integer $\pi \geqslant 1$.*

Proof. Suppose that $\beta > 1$ is non-integer, and X_β has a largest invariant measure. By Theorems 1.2 and 1.3, $\beta = \beta(B - 1 + \pi/(a\pi + 1))$ satisfies $\beta^{a\pi+1} = B\sum_{i=0}^{\pi} \beta^{ia}$, where either $B = 1$ and $a, \pi \geqslant 1$, or $a = 1$, $\pi \geqslant 1$, and $B \geqslant 2$.

If $a = 1$ then $\beta^{\pi+1} = B\sum_{i=0}^{\pi} \beta^i$ for some $\pi, B \geqslant 1$, and [14, Theorem 2] implies that β is Pisot. If $(a, B) = (2, 1)$ then $0 = \beta^{2\pi+1} - \sum_{i=0}^{\pi} \beta^{2i} = \beta^{2\pi+1} - (\beta^{2\pi+2} - 1)/(\beta^2 - 1)$ for some $\pi \geqslant 1$, so [36, p. 597] implies that β is Pisot.

The fourth smallest Pisot number is known to be $\theta_4 \approx 1.4655$, the dominant root of $\theta^3 - \theta^2 - 1$ (see [13, p. 120]). However, this cubic polynomial is $\beta^{a\pi+1} - B\sum_{i=0}^{\pi} \beta^{ia}$ for $(\pi, a, B) = (1, 2, 1)$, so in this case $\beta = \beta(1/3) = \theta_4$ is Pisot. If $a \geqslant 3$ then necessarily $B = 1$, so $B - 1 + \pi/(a\pi + 1) = \pi/(a\pi + 1) < 1/3$; then Lemma 3.12 gives $\beta = \beta(\pi/(a\pi + 1)) < \beta(1/3) = \theta_4$, and therefore $\beta = \beta(\pi/(a\pi + 1))$ is not Pisot unless it is one of the three smallest Pisot numbers.

It turns out that the three smallest Pisot numbers are all of the form $\beta(\pi/(a\pi + 1))$ for some $a \geqslant 3$ and $\pi \geqslant 1$. Siegel [36] proved that the smallest Pisot number is $\theta_1 \approx 1.3247$, the dominant root of $\theta^3 - \theta - 1$, and that the second smallest is $\theta_2 \approx 1.3802$, the dominant root of $\theta^4 - \theta^3 - 1$. Dufresnoy and Pisot [13, p. 120] proved that the third smallest Pisot number is $\theta_3 \approx 1.4432$, the dominant root of $\theta^5 - \theta^4 - \theta^3 + \theta^2 - 1$. If $(\pi, a) = (1, 4)$ then $\beta = \beta(\pi/(a\pi + 1)) = \beta(1/5)$ is the dominant root of $\beta^5 - \beta^4 - 1 = (\beta^2 - \beta + 1)(\beta^3 - \beta - 1)$, so $\beta = \beta(1/5) = \theta_1$ is Pisot. Similarly, if $(\pi, a) = (1, 3)$ then $\beta = \beta(1/4) = \theta_2$ is Pisot, while if $(\pi, a) = (2, 3)$ then $\beta = \beta(2/7) = \theta_3$ is Pisot. □

The following formula exists for the mass given to a cylinder set by a Sturmian measure.

LEMMA 3.13. *Let $\varrho \in [0,\infty)$. If $w_1 \ldots w_n \in \mathbb{N}_0^n$ and $c_k := \sum_{i=1}^k w_i$ for $k \in [0,n]$, then*

$$S_\varrho(\langle w_1 \ldots w_n \rangle_F) = \max\left(0,\ 1 + \min_{0\leqslant k\leqslant n}(c_k - k\varrho)\ -\ \max_{0\leqslant k\leqslant n}(c_k - k\varrho)\right). \tag{3}$$

Proof. The proof in [8, Proposition 1.2] is for $\varrho \in [0,1]$, and minor modifications yield (3) for $\varrho > 1$. □

REMARK 3.14. From Lemma 3.13, or alternatively by rearranging (2), we have

$$S_\varrho(\langle m_{X(\varrho)} \rangle_{X(\varrho)}) = \varrho - (m_{X(\varrho)} - 1)\,,$$

or, in other words,

$$S_\varrho(\langle m_{X(\varrho)} \rangle_{X(\varrho)}) = \begin{cases} \varrho - [\varrho] & \text{if } \varrho \in [0,\infty)\setminus\mathbb{N}_0, \\ 1 & \text{if } \varrho \in \mathbb{N}_0. \end{cases}$$

4. *Beta-shifts without a largest invariant measure*

The purpose of this section is to identify those beta-shifts that do not have a largest invariant measure. First, note that the Sturmian measure S_X is the only candidate for a largest element of $(\mathcal{M}_X, \prec)$.

LEMMA 4.1. *For $X \in \mathfrak{B}$, if $\mu(x, X^+]_X > S_X(x, X^+]_X$ for some $\mu \in \mathcal{M}_X$ and $x \in X$, then $(\mathcal{M}_X, \prec)$ has no largest element.*

Proof. By Lemma 3.1, if $\nu \in \mathcal{M}_X \setminus \{S_X\}$ then $\nu[X^-, X^+]_X < 1 = S_X[X^-, X]_X$. Therefore, by Lemma 2.10, S_X is the only possible largest element in $(\mathcal{M}_X, \prec)$. However, $\mu(x, X^+]_X > S_X(x, X^+]_X$, so S_X is, in fact, not the largest element in $(\mathcal{M}_X, \prec)$. □

LEMMA 4.2. *If $X \in \mathfrak{B}$ is not of Sturmian type, then $(\mathcal{M}_X, \prec)$ has no largest element.*

Proof. Since X is not of Sturmian type, and periodic points are dense in X (see [37]), there exists a periodic point x in the interval $(S_X^+, X^+]_X$, where S_X^+ denotes the largest point in the support of S_X. If μ denotes the unique invariant probability measure whose support equals $\{\sigma^i(x) : i \geqslant 0\}$, then $\mu(S_X^+, X^+]_X > 0 = S_X(S_X^+, X^+]_X$, so Lemma 4.1 implies that $(\mathcal{M}_X, \prec)$ has no largest element. □

In view of Lemma 4.2, we may now restrict attention to those beta-shifts that are of Sturmian type.

NOTATION 4.3. Define

$$U := \left\{\frac{\pi}{a\pi + 1} : a \in \mathbb{N}, \pi \in \mathbb{N}_0\right\} \cup \left\{A + \frac{\pi}{\pi+1} : A \in \mathbb{N}, \pi \in \mathbb{N}_0\right\}.$$

REMARK 4.4. In continued fraction notation,

$$U = \mathbb{N}_0 \cup \{[0; c_1, c_2] : c_1, c_2 \in \mathbb{N}\} \cup \{[c_0; 1, c_2] : c_0, c_2 \in \mathbb{N}\},$$

where

$$[c_0; c_1, c_2] := c_0 + \frac{1}{c_1 + \frac{1}{c_2}} \quad \text{for } c_0, c_1, c_2 \in \mathbb{N}.$$

The following is the main result of this section.

THEOREM 4.5. *If $\varrho \notin U$ then $(\mathcal{M}_{X(\varrho)}, \prec)$ has no largest element.*

To prove Theorem 4.5, it is convenient to define the set $V \supset U$ by

$$V := \left\{ A + \frac{\pi}{a\pi + 1} : A, \pi \in \mathbb{N}_0, a \in \mathbb{N} \right\}.$$

We shall see that, if $\varrho \in V \setminus U$, then the proof that $(\mathcal{M}_{X(\varrho)}, \prec)$ has no largest element is relatively straightforward. The heart of Theorem 4.5, therefore, is the case $\varrho \notin V$, and this will require considerable preparation. The strategy of the proof is as follows.

By Lemma 4.1, it is sufficient to find a measure $\mu \in \mathcal{M}_{X(\varrho)}$ that is not dominated by the Sturmian measure S_ϱ. Although such a μ is not unique, some care is needed in its selection; in particular, μ will depend on ϱ, so our first step will be to localize ϱ with respect to the points in V. To this end, for each $A \in \mathbb{N}_0$, we define

$$V_A := \left\{ A + \frac{\pi}{a\pi + 1} : a, \pi \in \mathbb{N} \right\}, \tag{4}$$

and note that

$$V = \bigcup_{A \in \mathbb{N}_0} V_A \cup \{A\}.$$

For $A \in \mathbb{N}_0$, we shall say that two elements $\varrho_1, \varrho_2 \in V_A$, with $\varrho_1 < \varrho_2$, are *consecutive* if $(\varrho_1, \varrho_2) \cap V_A = \emptyset$; equivalently, $\varrho_1 = [A; a, \pi]$ and $\varrho_2 = [A; a, \pi + 1]$ for some $a, \pi \in \mathbb{N}$. Now each $\varrho \notin V$ lies in between two consecutive rationals in V_A, for some $A \in \mathbb{N}_0$. Letting $\varrho_1 < \varrho_2$ denote these consecutive rationals, our next step will be to partition the open interval (ϱ_1, ϱ_2) into infinitely many sub-intervals $R_n(\varrho_1, \varrho_2)$, for $1 \leqslant n < \infty$ (see Definition 4.9). The choice of μ will depend on which of these sub-intervals contains ϱ. More precisely, for each $1 \leqslant n < \infty$ there is a shift-invariant probability measure $\mu = \mu_n$ (defined in Lemma 4.10) on F that belongs to $\mathcal{M}_{X(\varrho)}$ for every $\varrho \in R_n(\varrho_1, \varrho_2)$, and is not dominated by S_ϱ.

To prove that μ is not dominated by S_ϱ, it is convenient to first introduce the following notation.

NOTATION 4.6. For $\varrho \in [0, \infty)$ and $\varrho' \in [0, \infty) \cap \mathbb{Q}$, define

$$C(\varrho', \varrho) := \langle \zeta(\varrho') \rangle_{X(\varrho)},$$

the cylinder set in $X(\varrho)$ determined by $\zeta(\varrho')$, the periodic word of $X(\varrho')^+$.

For $\varrho \in V$, there is the following explicit formula for $\zeta(\varrho)$.

LEMMA 4.7. *If $\varrho = A + \frac{\pi}{a\pi+1}$, with $A \in \mathbb{N}_0$ and $a, \pi \in \mathbb{N}$, then $\zeta(\varrho)$ is the word*

$$(BA^{a-1})^{\pi-1} BA^a = (BA^{a-1})^\pi A\,, \tag{5}$$

where $B = A + 1$.

If $\varrho = A \in \mathbb{N}_0$ then $\zeta(\varrho) = A$.

Proof. If $\varrho = A \in \mathbb{N}$ then $X(\varrho)^+ = \overline{A}$, so $\zeta(\varrho) = A$. Now suppose that $\varrho = A + \pi/(a\pi + 1)$, with $A \in \mathbb{N}_0$ and $a, \pi \in \mathbb{N}$. The length of the word w defined by (5) is $a\pi + 1$, and its weight is $(a\pi + 1)A + \pi$, so its weight-per-symbol is $A + \pi/(a\pi + 1) = \varrho$. It is readily verified that $\overline{w}$ is balanced, so, by Proposition 3.5(vi), it is an atom of the Sturmian measure S_ϱ. Moreover, $\sigma^n(\overline{w}) \leqslant \overline{w}$ for all $n \geqslant 0$, so $X(\varrho)^+ = \overline{w}$. Therefore $\zeta(\varrho) = w$, because there is no finite word u such that $w = u^n$ for $n \geqslant 2$. □

We will show (Lemma 4.10 and Corollary 4.12) that the measure $\mu = \mu_n$ (defined in Lemma 4.10) gives greater mass to the cylinder set $C(\varrho_2, \varrho)$ than does S_ϱ. By Lemma 2.10, this implies that S_ϱ does not dominate μ, since, by the following result, $C(\varrho_2, \varrho)$ is an upper interval in $X(\varrho)$.

Lemma 4.8. *Let $A \in \mathbb{N}_0$, and suppose that $\varrho_1 < \varrho_2$ are consecutive elements in V_A. For each $\varrho \in (\varrho_1, \varrho_2)$, the cylinder set $C(\varrho_2, \varrho) = \langle \zeta(\varrho_2) \rangle_{X(\varrho)}$ is an upper interval in the beta-shift $X(\varrho)$.*

Proof. The proof is immediate from Lemma 3.9. □

We can now define the sub-intervals $R_n(\varrho_1, \varrho_2)$ alluded to previously.

Definition 4.9. Suppose that $\varrho_1 < \varrho_2$ are consecutive elements in V_A, for some $A \in \mathbb{N}_0$, where $\varrho_1 = p_1/q_1$, $\varrho_2 = p_2/q_2$, and $p_1, p_2, q_1, q_2 \in \mathbb{N}$, with $\gcd(p_1, q_1) = 1 = \gcd(p_2, q_2)$. For $n \in \mathbb{N}_0$, define

$$r_n(\varrho_1, \varrho_2) := \frac{p_2 q_1 n + p_1}{q_2 q_1 n + q_1},$$

and note that

$$\zeta(r_n(\varrho_1, \varrho_2)) = \zeta(\varrho_2)^{q_1 n} \zeta(\varrho_1).$$

Define

$$R_1(\varrho_1, \varrho_2) := \big(r_0(\varrho_1, \varrho_2), r_1(\varrho_1, \varrho_2)\big) = (\varrho_1, r_1(\varrho_1, \varrho_2))$$

and

$$R_n(\varrho_1, \varrho_2) := \big[r_{n-1}(\varrho_1, \varrho_2), r_n(\varrho_1, \varrho_2)\big) \quad \text{for } n \geqslant 2.$$

Our strategy is to show that, for each $n \geqslant 1$, if $\varrho \in R_n(\varrho_1, \varrho_2)$ then the invariant measure μ supported on the periodic orbit defined by the word $\zeta(\varrho_2)^n A$ lies in $\mathcal{M}_{X(\varrho)}$, and gives larger mass to the upper interval $C(\varrho_2, \varrho)$ than does the Sturmian measure S_ϱ. The following is the first step.

Lemma 4.10. *Let $A \in \mathbb{N}_0$, and suppose that $\varrho_1 < \varrho_2$ are consecutive elements in V_A. For all $n \geqslant 1$ and every $\varrho \in R_n(\varrho_1, \varrho_2)$, the periodic orbit defined by $\zeta(\varrho_2)^n A$ lies in $X(\varrho)$; consequently, the invariant measure μ_n carried by this periodic orbit belongs to $\mathcal{M}_{X(\varrho)}$.*

If $\varrho_2 = p_2/q_2$, where $p_2, q_2 \in \mathbb{N}$ and $\gcd(p_2, q_2) = 1$, then

$$\mu_n(C(\varrho_2, \varrho)) = \frac{n}{nq_2 + 1}\,. \tag{6}$$

Proof. Let $\varrho_1 = p_1/q_1$, where $p_1, q_1 \in \mathbb{N}$ and $\gcd(p_1, q_1) = 1$, and suppose that $\varrho \in R_n(\varrho_1, \varrho_2)$.

First consider the case $n = 1$, so that $R_n(\varrho_1, \varrho_2)$ is the *open* interval $(p_1/q_1, (p_2q_1 + p_1)/(q_2q_1 + q_1))$. If $\varrho \in R_1(\varrho_1, \varrho_2)$ then, by Lemma 3.9,

$$X(\varrho)^+ > \zeta(\varrho_2)\overline{\zeta(\varrho_1)} \,. \tag{7}$$

Moreover,

$$\overline{\zeta(\varrho_2)^n A} = \overline{\zeta(\varrho_2)A} < \zeta(\varrho_2)\overline{\zeta(\varrho_1)} \,, \tag{8}$$

because

$$\left(\overline{\zeta(\varrho_2)A}\right)_i = \left(\zeta(\varrho_2)\overline{\zeta(\varrho_1)}\right)_i \quad \text{for } 1 \leqslant i \leqslant q_2 \,,$$

and Lemma 4.7 gives

$$\left(\overline{\zeta(\varrho_2)A}\right)_{q_2+1} = A < B = \left(\zeta(\varrho_2)\overline{\zeta(\varrho_1)}\right)_{q_2+1} .$$

Combining (7) and (8) gives

$$\overline{\zeta(\varrho_2)^n A} < X(\varrho)^+ . \tag{9}$$

However, $\overline{\zeta(\varrho_2)^n A}$ is the largest point in its orbit, so (9) and Lemma 2.15 together imply that this orbit is contained in $X(\varrho)$, as required.

Now suppose that $n \geqslant 2$. It is readily verified that $X(r_{n-1}(\varrho_1, \varrho_2))^+ = \overline{\zeta(\varrho_2)^{q_1(n-1)}\zeta(\varrho_1)}$. Since $\varrho \geqslant r_{n-1}(\varrho_1, \varrho_2)$, Lemma 3.12 gives

$$X(\varrho)^+ \geqslant X(r_{n-1}(\varrho_1, \varrho_2))^+ = \overline{\zeta(\varrho_2)^{q_1(n-1)}\zeta(\varrho_1)} \,. \tag{10}$$

We claim that

$$\overline{\zeta(\varrho_2)^n A} < \overline{\zeta(\varrho_2)^{q_1(n-1)}\zeta(\varrho_1)}. \tag{11}$$

To prove (11), note that $\varrho_1 = p_1/q_1$ is not an integer, so $q_1 \geqslant 2$, and therefore $n \leqslant q_1(n-1)$. Hence the first nq_2 symbols of $\overline{\zeta(\varrho_2)^{q_1(n-1)}\zeta(\varrho_1)}$ coincide with the first nq_2 symbols of $\overline{\zeta(\varrho_2)^n A}$. Now the $(nq_2 + 1)$th symbol of $\overline{\zeta(\varrho_2)^{q_1(n-1)}\zeta(\varrho_1)}$ is the first symbol of either $\zeta(\varrho_2)$ or $\zeta(\varrho_1)$, namely $B = A + 1$ (by Lemma 4.7); thus

$$(\overline{\zeta(\varrho_2)^{q_1(n-1)}\zeta(\varrho_1)})_{nq_2+1} = A + 1 > A = (\overline{\zeta(\varrho_2)^n A})_{nq_2+1},$$

so indeed (11) holds. Combining this with (10) gives $\overline{\zeta(\varrho_2)^n A} < X(\varrho)^+$, and, because $\overline{\zeta(\varrho_2)^n A}$ is the largest point in its orbit, Lemma 2.15 implies that the whole of this orbit lies in $X(\varrho)$, as required.

To prove (6) for $n \geqslant 1$ we must show that $\{1 \leqslant i \leqslant nq_2 + 1 : \pi_{q_2}(\sigma^{i-1}x) = \zeta(\varrho_2)\}$ has cardinality n, where $x := \overline{\zeta(\varrho_2)^n A}$. For this it suffices to check that $\zeta(\varrho_2)$ is not a sub-word of either $\alpha(\varrho_2)\omega(\varrho_2)$ or $\alpha(\varrho_2)A\omega(\varrho_2)$, where $\alpha(\varrho_2)$ (respectively, $\omega(\varrho_2)$) denotes the length-$(q_2 - 1)$ word obtained by deleting the first (respectively, the last) symbol from the word $\zeta(\varrho_2)$; these facts are readily verified using the formula (5) from Lemma 4.7. □

Lemma 4.11. *Suppose that $A \in \mathbb{N}_0$ and $\varrho_1 = p_1/q_1 < \varrho_2 = p_2/q_2$ are consecutive elements in V_A, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$ and $\gcd(p_1, q_1) = 1 = \gcd(p_2, q_2)$. Then, for all $\varrho \in [\varrho_1, \varrho_2]$, we have*

$$S_\varrho(C(\varrho_2, \varrho)) = q_1\varrho - p_1 \,, \tag{12}$$

and, in particular, $\varrho \mapsto S_\varrho(C(\varrho_2, \varrho))$ is strictly increasing on $[\varrho_1, \varrho_2]$.

Proof. Write $\varrho_1 = A + \pi/(a\pi + 1)$ and $\varrho_2 = A + (\pi + 1)/(a(\pi + 1) + 1)$ for $a, \pi \in \mathbb{N}$. In view of Lemma 3.13, we wish to evaluate, for $\varrho \in [\varrho_1, \varrho_2]$, the expression

$$1 + \min_{0 \leqslant k \leqslant q_2} (c_k - k\varrho) - \max_{0 \leqslant k \leqslant q_2} (c_k - k\varrho),$$

where

$$c_k := \sum_{i=1}^{k} \zeta(\varrho_2)_i \quad \text{for } k \in [0, q_2]\,.$$

By Lemma 4.7, $\zeta(\varrho_2) = (BA^{a-1})^{\pi} A$, where $B = A + 1$. Therefore $c_0 = 0$,

$$c_k = kA + i \quad \text{for } k \in [(i-1)a + 1, ia],\ i \in [1, \pi + 1], \tag{13}$$

and

$$c_{q_2} = c_{(\pi+1)a+1} = ((\pi + 1)a + 1)A + \pi + 1. \tag{14}$$

From (13) and (14) we see that, for $k \in [0, q_2]$, the minimum value of $c_k - k\varrho$ is attained when $k = ja$ for some $0 \leqslant j \leqslant \pi$. Therefore

$$\min_{0 \leqslant k \leqslant q_2} (c_k - k\varrho) = \min_{0 \leqslant j \leqslant \pi} j(aA + 1 - a\varrho).$$

However, $aA + 1 - a\varrho > 0$ because $\varrho \leqslant \varrho_2 = A + (\pi + 1)/(a(\pi + 1) + 1) < A + 1/a$, so the minimum is attained when $j = 0$; thus

$$\min_{0 \leqslant k \leqslant q_2} (c_k - k\varrho) = 0\,. \tag{15}$$

The maximum value of $c_k - k\varrho$ is attained when $k = ja + 1$ for some $0 \leqslant j \leqslant \pi$, so

$$\max_{0 \leqslant k \leqslant q_2} (c_k - k\varrho) = \max_{0 \leqslant j \leqslant \pi} j(aA + 1 - a\varrho) + A + 1 - \varrho = \pi(aA + 1 - a\varrho) + A + 1 - \varrho\,, \tag{16}$$

again because $aA + 1 - a\varrho > 0$. From (15) and (16), and since $p_1/q_1 = A + \pi/(a\pi + 1)$, we have

$$1 + \min_{0 \leqslant k \leqslant q_2} (c_k - k\varrho) - \max_{0 \leqslant k \leqslant q_2} (c_k - k\varrho) = \varrho(q\pi + 1) - ((a\pi + 1)A + \pi) = q_1\varrho - p_1.$$

In particular, $q_1\varrho - p_1 \geqslant 0$ for $\varrho \in [\varrho_1, \varrho_2]$, so (12) follows from Lemma 3.13. □

COROLLARY 4.12. *Suppose that $A \in \mathbb{N}_0$ and $\varrho_1 = p_1/q_1 < \varrho_2 = p_2/q_2$ are consecutive elements in V_A, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$ and $\gcd(p_1, q_1) = 1 = \gcd(p_2, q_2)$. For all $n \geqslant 1$ and $\varrho \in R_n(\varrho_1, \varrho_2)$, we have*

$$S_\varrho(C(\varrho_2, \varrho)) < S_{r_n(\varrho_1,\varrho_2)}(C(\varrho_2, r_n(\varrho_1, \varrho_2))) = \frac{n}{nq_2 + 1} = \mu_n(C(\varrho_2, \varrho)). \tag{17}$$

Proof. By Lemma 4.11, the function $\varrho \mapsto S_\varrho(C(\varrho_2, \varrho))$ is strictly increasing on the interval $[\varrho_1, \varrho_2]$, and hence on each interval $R_n(\varrho_1, \varrho_2)$ for $n \geqslant 1$. Therefore, if $\varrho \in R_n(\varrho_1, \varrho_2)$ then $S_\varrho(C(\varrho_2, \varrho)) < S_{r_n(\varrho_1,\varrho_2)}(C(\varrho_2, r_n(\varrho_1, \varrho_2)))$. After setting $\varrho = r_n(\varrho_1, \varrho_2) = (p_2q_1n + p_1)/(q_2q_1n + q_1)$ in (12), a short calculation gives $S_{r_n(\varrho_1,\varrho_2)}(C(\varrho_2, r_n(\varrho_1, \varrho_2))) = n/(nq_2 + 1)$, and by (6) this is also the value of $\mu_n(C(\varrho_2, \varrho))$, so (17) is proved. □

Proof of Theorem 4.5. Each $\varrho \notin V$ belongs to (ϱ_1, ϱ_2), where $\varrho_1 < \varrho_2$ are consecutive elements in V_A for some $A \in \mathbb{N}_0$. Now $C(\varrho, \varrho_2)$ is an upper interval in $X(\varrho)$ by Lemma 4.8, so the inequality (17) implies that S_ϱ does not dominate $\mu_n \in \mathcal{M}_{X(\varrho)}$. Therefore, by Lemma 4.1, $(\mathcal{M}_{X(\varrho)}, \prec)$ has no largest element.

It remains to show that if $\varrho \in V \setminus U$ then S_ϱ does not dominate every measure in $\mathcal{M}_{X(\varrho)}$. Now $\varrho \in V \setminus U$ implies first that $q = a\pi + 1$ for $a \geqslant 2$, and secondly that $\varrho > 1$, and hence $m = m_{X(\varrho)} \geqslant 2$. Now $X(\varrho)^+$ is a (Sturmian) sequence on the symbol set $\{m-1, m\}$ and belongs to $\langle m \rangle_{X(\varrho)}$, so both points $\overline{0m}$ and $\overline{m0}$ are smaller than $X(\varrho)^+$. Hence the period-2 orbit $\{\overline{0m}, \overline{m0}\}$ lies in $X(\varrho)$, the invariant measure μ supported by this orbit lies in $\mathcal{M}_{X(\varrho)}$, and clearly $\mu(\langle m\rangle_{X(\varrho)}) = 1/2$. However, by Remark 3.14, $S_\varrho(\langle m\rangle_{X(\varrho)}) = \{\varrho\} = \pi/(a\pi+1) < 1/2$. Therefore S_ϱ does not dominate μ, and the proof is complete. □

5. *Beta-shifts with a largest invariant measure*

Having identified, in Section 4, certain Sturmian-type beta-shifts that do not have a largest invariant measure, we now show that all other Sturmian-type beta-shifts do have a largest invariant measure.

First, the singleton beta-shift $\{\overline{0}\}$ is easily dealt with: it supports a unique (invariant) probability measure $\delta_{\overline{0}}$, which, of course, is its largest invariant measure. Therefore, we shall henceforth assume that $X \in \mathfrak{B} \setminus \{\overline{0}\}$.

NOTATION 5.1. Let $\varrho = A + \pi/q$ with $A, \pi \in \mathbb{N}_0$, $q \in \mathbb{N}$, and $\gcd(\pi, q) = 1$. Let $s_1 < \ldots < s_q$ denote the atoms of $S_{X(\varrho)}$, and define

$$J_0 := [\overline{0}, s_1]_{X(\varrho)} \quad \text{and} \quad J_j := (s_j, s_{j+1}]_{X(\varrho)} \text{ for } 1 \leqslant j \leqslant q-1\,.$$

The collection $\{J_i\}_{i=0}^{q-1}$ will be called the *Sturmian partition* of $X(\varrho)$.

With the notation as above, $J_1 \cup \ldots \cup J_{q-\pi-1} \subset \langle m_{X(\varrho)} - 1\rangle_{X(\varrho)}$ and $J_{q-\pi} \cup \ldots \cup J_{q-1} = \langle m_{X(\varrho)}\rangle_{X(\varrho)}$.

NOTATION 5.2. Henceforth it will be notationally convenient to denote $[x, x']_X$ by $[x, x']$, and $(x, x']_X$ by $(x, x']$.

The following Lemma 5.3 records the mapping properties of the Sturmian partition of $X(\varrho)$ under the shift map $\sigma : X(\varrho) \to X(\varrho)$. A *preimage* always means a preimage under $\sigma|_{X(\varrho)}$; so to say that x has l preimages means that $(\sigma|_{X(\varrho)})^{-1}(x)$ has cardinality l. The key feature is that the last π intervals $J_{q-\pi}, \ldots, J_{q-1}$ are mapped onto the first π intervals $J_0, \ldots, J_{\pi-1}$; thus points in these first π intervals have one more shift preimage than other points in $X(\varrho)$.

LEMMA 5.3. *For $\varrho = A + \pi/q$ with $A, \pi \in \mathbb{N}_0$, $q \in \mathbb{N}$, and $\gcd(\pi, q) = 1$, let $J_0, \ldots, J_{q-1}$ be the Sturmian partition of $X(\varrho)$.*

(i) *Each $x \in J_0 \cup \ldots \cup J_{\pi-1}$ has $m_{X(\varrho)} + 1$ preimages, of which $m_{X(\varrho)}$ lie in J_0, and one lies in $X(\varrho) \setminus J_0$. Each $x \in J_n$, for $n \in [0, \pi-1]$, has a preimage in $J_{n+q-\pi}$.*

(ii) *Each $x \in J_\pi$ has $m_{X(\varrho)}$ preimages, all of which lie in J_0.*

(iii) *Each $x \in J_{\pi+1} \cup \ldots \cup J_{q-1}$ has $m_{X(\varrho)}$ preimages, of which $m_{X(\varrho)} - 1$ lie in J_0, and one lies in $X(\varrho) \setminus J_0$. Each $x \in J_n$, for $n \in [\pi+1, q-1]$, has a preimage in $J_{n-\pi}$.*

Proof. Let us denote $X(\varrho)$ by X, and $m_{X(\varrho)}$ by m. Points in $J_0 \cup \ldots \cup J_{\pi-1} = [\overline{0}, \sigma(X^+)]$ have precisely $m+1$ preimages, and points in $J_\pi \cup \ldots \cup J_{q-1} = (\sigma(X^+), X^+]$ have precisely m preimages (cf. Remark 2.17).

(i) Let $n \in [0, \pi-1]$. Now $J_n = (s_n, s_{n+1}]$, so

$$mJ_n = (s_{n+q-\pi}, s_{n+1+q-\pi}] = J_{n+q-\pi} \subset J_1 \cup \ldots \cup J_{q-\pi-1} \subset X \setminus J_0;$$

thus each $x \in J_0 \cup \ldots \cup J_{\pi-1}$ has at least one preimage in $X \setminus J_0$. However, x has at most one preimage in $X \setminus J_0$, since $X \setminus J_0$ is a proper sub-interval of $[X^-, X^+]$, so each $x \in J_0 \cup \ldots \cup J_{\pi-1}$ has exactly one preimage in $X \setminus J_0$, and hence exactly m preimages in J_0.

(ii) If $\varrho \in \mathbb{N}_0$ then $\pi = 0$ and $m = \varrho$; thus $J_\pi = J_0 = [\overline{0}, s_1] = [\overline{0}, X^+] = X = F_\varrho$; thus every point in J_π has precisely $\varrho = m$ preimages in $X = J_0$, as required. If $\varrho \notin \mathbb{N}_0$ then $\pi \geqslant 1$ and $J_\pi = (s_\pi, s_{\pi+1}]$, so $(m-1)J_\pi = ((m-1)s_\pi, s_1] \subset J_0$. Moreover, if $x \in J_\pi$ and $k \in [0, m)$ then $kx < (m-1)x$, so $kx \in J_0$. Therefore, if $x \in J_\pi$ then all m of its preimages lie in J_0.

(iii) Let $n \in [\pi+1, q-1]$. Now $J_n = (s_n, s_{n+1}]$, so

$$(m-1)J_n = (s_{n-\pi}, s_{n+1-\pi}] = J_{n-\pi} \subset J_1 \cup \ldots \cup J_{q-\pi-1} \subset X \setminus J_0;$$

thus each $x \in J_{\pi+1} \cup \ldots \cup J_{q-1}$ has at least one preimage in $X \setminus J_0$. However, x has at most one preimage in $X \setminus J_0$, since $X \setminus J_0$ is a proper sub-interval of $[X^-, X^+]$, so each $x \in J_{\pi+1} \cup \ldots \cup J_{q-1}$ has exactly one preimage in $X \setminus J_0$, and hence exactly $m-1$ preimages in J_0. □

THEOREM 5.4. *If $\varrho \in U$ then $(\mathcal{M}_{X(\varrho)}, \prec)$ has a largest element, namely its Sturmian measure $S_{X(\varrho)} = S_\varrho$.*

Proof. Let $X = X(\varrho)$, where $\varrho \in U$. Hence $\varrho = A + \pi/q$, where $\pi \in \mathbb{N}_0$ and either $A \in \mathbb{N}$ and $q = \pi + 1$, or $A = 0$ and $q = a\pi + 1$ for some $a \in \mathbb{N}$. We must show that $\mu \prec S_X$ for every $\mu \in \mathcal{M}_X$. By Lemma 2.10, this is true if and only if $\mu(x, X^+] \leqslant S_X(x, X^+]$ for all $x \in X$. However, S_X is purely atomic, with atoms $s_1 < \ldots < s_q$, so it is enough to show that

$$\mu(s_{q-i}, X^+] \leqslant S_X(s_{q-i}, X^+] \tag{18}$$

for all $1 \leqslant i \leqslant q-1$. In other words, we wish to show that S_X is a measure of maximum hitting frequency (cf. Remark 2.12(ii)) for each interval $(s_{q-i}, X^+] = J_{q-i} \cup \ldots \cup J_{q-1}$, for $1 \leqslant i \leqslant q-1$. The main substance of our argument will be to first establish (18) for $i \in [1, \pi]$. Having proved this, we will then explain how the case of general $1 \leqslant i \leqslant q$ follows.

Therefore, for each $i \in [1, \pi]$, define

$$f_i := \chi_{(s_{q-i}, s_q]}.$$

Define $\varphi_i : X \to \mathbb{R}$ to be constant on each interval J_j of the Sturmian partition, with

$$\varphi_i|_{J_0} := 0,$$

and, for $l \in [0, a-1]$, with

$$\varphi_i|_{J_{k+l\pi}} := \begin{cases} \frac{i}{q}(ak-l) & \text{for } k \in [1, \pi-(i+1)], \\ \frac{i}{q}(ak-l) + \pi - k - i & \text{for } k \in [\pi - i, \pi]. \end{cases} \tag{19}$$

Note that, in fact, for $l \in [0, a-1]$, we have

$$\varphi_i|_{J_{k+l\pi}} = \frac{i}{q}(ak-l) \quad \text{for all } k \in [1, \pi - i]. \tag{20}$$

In order to prove (18) we first claim that

$$\max_{y \in \sigma^{-1}(x)} (f_i + \varphi_i)(y) = \varphi_i(x) + \frac{i}{q} \quad \text{for all } x \in X. \tag{21}$$

For each $x \in X$, let $\tau(x)$ denote its unique preimage in the interval $(X^-, X^+]$; that is, $\tau(x) = m_X x$ for $x \in [\overline{0}, \sigma(X^+)]$ and $\tau(x) = (m_X - 1)x$ for $x \in (\sigma(X^+), X^+]$ (cf. Remark 2.17). As a step toward establishing (21) we shall first prove that, for all $x \in X$, we have

$$\max_{y \in \sigma^{-1}x} (f_i + \varphi_i)(y) = (f_i + \varphi_i)(\tau(x)). \tag{22}$$

Note that (22) is trivially true if $x \in J_\pi$, since all of its preimages lie in J_0 by Lemma 5.3(ii), and hence $f_i + \varphi_i$ vanishes at each such preimage.

Now suppose that $\varrho = A + \pi/q$, where $A, \pi \in \mathbb{N}_0$ and $q = a\pi + 1$ for $a \in \mathbb{N}$; at this stage we do not require the fact that if $A > 0$ then $a = 1$. Every $x \in J_0 \cup \ldots \cup J_{\pi-1}$ has, by Lemma 5.3(i), at least one preimage in J_0, and precisely one preimage in $X \setminus J_0$, namely $\tau(x)$. Since $(f_i + \varphi_i)|_{J_0} = 0$, to establish (22) it suffices to prove that $(f_i + \varphi_i)(\tau(x)) > 0$. For this, let $n \in [0, \pi - 1]$ be such that $x \in J_n$, so that $\tau(x) \in J_{n+q-\pi}$ by Lemma 5.3(i). We wish to show that

$$(f_i + \varphi_i)|_{J_{n+q-\pi}} > 0. \tag{23}$$

First, if $n \in [0, \pi - (i+1)]$ then $n + q - \pi \in [q - \pi, q - (i+1)] \subset [0, q - (i+1)]$, so $f_i|_{J_{n+q-\pi}} = 0$, and therefore (19) and (20) give

$$(f_i + \varphi_i)|_{J_{n+q-\pi}} = \varphi_i|_{J_{n+q-\pi}} = \varphi_i|_{J_{(n+1)+(a-1)\pi}} = \frac{i}{q}(an+1) > 0. \tag{24}$$

Secondly, if $n \in [\pi - i, \pi - 1]$ then $n + q - \pi \in [q - i, q - 1]$, so $f_i|_{J_{n+q-\pi}} = 1$, and therefore

$$\begin{aligned}(f_i + \varphi_i)|_{J_{n+q-\pi}} &= 1 + \varphi_i|_{J_{n+q-\pi}} \\ &= 1 + \varphi_i|_{J_{(n+1)+(a-1)\pi}} \\ &= \frac{i}{q}(an+1) + \pi - n - i \\ &= \left(\frac{ai}{q} - 1\right) n + \pi - i + \frac{i}{q}. \end{aligned} \tag{25}$$

Expression (25) is decreasing in n, because $ai/q \leqslant a\pi/q < 1$, so for $n \in [\pi - i, \pi - 1]$ it attains its minimum when $n = \pi - 1$; that is,

$$(f_i + \varphi_i)|_{J_{n+q-\pi}} \geqslant \left(\frac{ai}{q} - 1\right)(\pi - 1) + \pi - i + \frac{i}{q} = 1 - \frac{ai}{q} \geqslant 1 - \frac{a\pi}{q} > 0\,. \tag{26}$$

Combining (24) and (26) yields (23), and hence (22) holds for all $x \in J_0 \cup \ldots \cup J_{\pi-1}$.

If $\varrho \in [1, \infty) \cap U$ then $q = \pi + 1$, so the preceding argument establishes (22) for all $x \in J_0 \cup \ldots \cup J_{q-1} = X$, as required. It therefore remains to check that (22) holds when $\varrho \in [0, 1) \cap U$ and $x \in J_{\pi+1} \cup \ldots \cup J_{q-1}$. However, in this case $m_X = 1$, so Lemma 5.3(iii) implies that x has a single preimage, namely $\tau(x)$, and thus (22) is trivially true. Therefore, for all $\varrho \in U$, we have established (22) for all $x \in X$.

We shall now derive (21) from (22) by checking that, for all $x \in X$, we have

$$(f_i + \varphi_i)(\tau(x)) = \varphi_i(x) + \frac{i}{q}. \tag{27}$$

If $x \in J_\pi$ then $(f_i + \varphi_i)(\tau(x)) = 0$ since $\tau(x) \in J_0$, and $\varphi_i(x) = -i/q$ (with $k = \pi$ and $l = 0$ in (19)), so (27) holds. If $x \in J_n$ for $n \in [0, \pi - (i+1)]$ then, again by (19), $\varphi_i(x) = ian/q$; thus $\varphi_i(x) + i/q = i(an+1)/q = (f_i + \varphi_i)(\tau(x))$ by (24), so (27) holds. If $x \in J_n$ for $n \in [\pi - i, \pi - 1]$ then, by (19), $\varphi_i(x) = ian/q + \pi - n - i$; thus $\varphi_i(x) + i/q = i(an+1)/q + \pi - n - i = (f_i + \varphi_i)(\tau(x))$ by (25), so (27) holds. As before, if $\varrho \in [1, \infty) \cap U$ then $q = \pi + 1$, so (27) holds for all $x \in X$. Finally, if $\varrho \in [0, 1) \cap U$ (that is, $m_X = 1$) and $x \in J_{\pi+1} \cup \ldots \cup J_{q-1}$ then $x \in J_{k+l\pi}$ for some $k \in [1, \pi]$ and $l \in [1, a-1]$. Now, by Lemma 5.3(iii), $\tau(x) \in J_{k+(l-1)\pi}$, so $f_i(\tau(x)) = 0$; therefore $(f_i + \varphi_i)(\tau(x)) = \varphi_i(\tau(x)) = \varphi_i|_{J_{k+(l-1)\pi}} = \varphi_i(x) + i/q$, so (27) holds. We have therefore proved that (27) holds for all $x \in X$, and combined with (22) this yields (21).

We can now derive (18) from (21) as follows. Replacing x by $\sigma(x)$ in (21) yields

$$(f_i + \varphi_i)(x) \leqslant \max_{y \in \sigma^{-1}\sigma(x)} (f_i + \varphi_i)(y) = \varphi_i(\sigma(x)) + \frac{i}{q} \quad \text{for all } x \in X,$$

so the function $f_i + \varphi_i - \varphi_i \circ \sigma$ is bounded above by i/q. Therefore, for all $\mu \in \mathcal{M}_X$, we have

$$\mu(s_{q-i}, X^+] = \mu(f_i) = \mu(f_i + \varphi_i - \varphi_i \circ \sigma) \leqslant \frac{i}{q}.$$

However, $S_X(s_{q-i}, X^+] = i/q$, so (18) follows in the case $i \in [1, \pi]$.

We now show how to deduce (18) for general $i \in [1, q-1]$. Writing $i = b\pi + j$ for $b \in [0, a-1]$ and $j \in [1, \pi]$, we first claim that

$$\mu(s_{q-(b\pi+j)}, s_{q-b\pi}] \leqslant S_X(s_{q-(b\pi+j)}, s_{q-b\pi}] \quad \text{for all } \mu \in \mathcal{M}_X. \tag{28}$$

The proof of (28) will be via induction on b; note that the base case $b = 0$ with $j \in [1, \pi]$, has already been established (it is precisely (18) for $i \in [1, \pi]$). To prove (28) for $b \in [1, a-1]$ and $j \in [1, \pi]$, note that $\sigma((s_{q-(b\pi+j)}, s_{q-b\pi}]) = (s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}]$, so

$$(s_{q-(b\pi+j)}, s_{q-b\pi}] \subset \sigma^{-1}(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}].$$

Therefore, if $\mu \in \mathcal{M}_X$ then

$$\mu(s_{q-(b\pi+j)}, s_{q-b\pi}] \leqslant \mu(\sigma^{-1}(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}]) = \mu(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}]. \tag{29}$$

However,

$$\sigma^{-1}(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}] = (s_{q-(b\pi+j)}, s_{q-b\pi}] \cup Y,$$

where $Y \subset J_0 \subset [\overline{0}, X^-]$, and the support of S_X is contained in $(X^-, X^+]$ (cf. Remark 3.8), so

$$S_X(s_{q-(b\pi+j)}, s_{q-b\pi}] = S_X(\sigma^{-1}(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}]) = S_X(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}]. \tag{30}$$

Combining the inductive hypothesis

$$\mu(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}] \leqslant S_X(s_{q-((b-1)\pi+j)}, s_{q-(b-1)\pi}] \tag{31}$$

with (29) and (30) yields (28), as claimed.

We now use (28) to deduce (18), that is, that S_X is a measure of maximum hitting frequency for the interval $(s_{q-i}, s_q]$, in the remaining cases $i \in [\pi+1, q-1]$. Writing $i = c\pi + \kappa$, where $\kappa \in [1, \pi]$ and $c \in [1, a-1]$, we can write $(s_{q-i}, s_q]$ as the disjoint union

$$(s_{q-i}, s_q] = (s_{q-(c\pi+\kappa)}, s_{q-c\pi}] \cup \bigcup_{\lambda=1}^{c} (s_{q-\lambda\pi}, s_{q-(\lambda-1)\pi}]. \tag{32}$$

Now S_X is a measure of maximum hitting frequency for each of the $c+1$ intervals in the disjoint union (32), using (28) in the cases $(b, j) = (\lambda - 1, \pi)$ for $\lambda \in [1, c]$, and in the case $(b, j) = (c, \kappa)$. Consequently, S_X is also a measure of maximum hitting frequency for the interval $(s_{q-i}, s_q]$, as required. □

REMARK 5.5. The proof of Theorem 5.4 via the introduction of the function φ_i is inspired by ideas of Bousch [**7**] in a related setting, in particular, his notion of the *Sturmian condition* (cf. also [**3**, **16**]). The usefulness in ergodic optimization of solutions φ_i to equations of the form (21) had been observed by Conze and Guivarc'h [**11**].

It is now a simple matter to complete the theorems presented in Section 1.

Proof of Theorems 1.1–1.3. In view of Theorems 4.5 and 5.4, to conclude the proof of Theorems 1.1–1.3 it only remains to check the equivalence of statements (ii) and (iii) in Theorems 1.2 and 1.3. Let X denote the beta-shift whose largest element X^+ is the periodic sequence with periodic word $(10^{a-1})^\pi 0$ for $a, \pi \geqslant 1$ (respectively, $B^\pi(B-1)$ for some $\pi \geqslant 1$ and $B \geqslant 2$). By [**6**, Prop. 2.3 (2)], $X = X_\beta$ if and only if $1 = \sum_{i=1}^{\infty} (X^+)_i \beta^{-i}$, and this equation is easily seen to be equivalent to $\zeta^{a\pi+1} - \sum_{i=0}^{\pi} \zeta^{ia} = 0$ (respectively, $\zeta^{\pi+1} - B\sum_{i=0}^{\pi} \zeta^i = 0$). □

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