# DECOMPOSITION OF THE CONJUGACY REPRESENTATION FOR SYMMETRIC GROUPS AND SUBGROUP GROWTH

Thomas W. Müller and Jan-Christoph Schlage-Puchta

Abstract. We obtain estimates for the multiplicities of the conjugacy representation of symmetric groups complementing recent results of Roichman. These estimates in turn are used to compute the subgroup growth for a substantial class of large groups including in particular all Fuchsian groups as well as one-relator groups whose relator involves iterated commutators.

## 1. Introduction and Main Results

For a partition $\lambda \vdash n$ denote by $m_\lambda$ the multiplicity of the irreducible character $\chi_\lambda$ in the character of the conjugacy representation. In [7] Roichman shows that $m_\lambda$ is close to $\chi_\lambda(1)$ provided that $\lambda_1$ (the first part of $\lambda$) satisfies $\lambda_1 \leq \delta n$ for some sufficiently small $\delta$. Moreover, he proves that $\frac{m_\lambda}{\chi_\lambda(1)}$ is bounded and approximately computable if $\frac{\lambda_1}{n}$ is bounded away from 1. This leaves open the case where $\lambda_1$ can get arbitrarily close to $n$. Here, we show the following, including Roichman's results for the sake of easy reference.

**Theorem 1.** *Let $\lambda \vdash n$ be a partition, $\chi_\lambda$ the corresponding character, and let $m_\lambda$ be the multiplicity of $\chi_\lambda$ in the conjugacy representation of $S_n$. Suppose that $\lambda_1' \leq \lambda_1$.*

(i) *If $\frac{\lambda_1}{n} \to 0$, then $m_\lambda \sim \chi_\lambda(1)$.*
(ii) *If $\frac{\lambda_1}{n} \leq 1-\delta$, then $m_\lambda \ll_\delta \chi_\lambda(1)$.*
(iii) *If $\sqrt{n} < n-\lambda_1 < n/2$, then $m_\lambda < e^{c\sqrt{\frac{n}{n-\lambda_1}}}\chi_\lambda(1)$ with some $c>0$.*
(iv) *If $n-\lambda_1 < c_1\sqrt{n}$, then $m_\lambda < p(n)e^{-c_2(n-\lambda_1)}\chi_\lambda(1)$, where $c_2$ is a positive constant, depending on $c_1$.*
(v) *If $n-\lambda < n^{1/4}/2$, then*

$$m_\lambda < p(n)\chi_\lambda(1)\left(\frac{(n-\lambda_1)^4}{n}\right)^{(n-\lambda_1)/2}.$$

(vi) *If $d = n-\lambda_1$ is fixed, then*

$$m_\lambda = \big(1+\mathcal{O}(n^{-1/2})\big)d!\Big(\frac{\sqrt{6}}{\pi}\Big)^d\chi_\mu(1)n^{d/2},$$

*where $\mu \vdash n-\lambda_1$ is the partition obtained from $\lambda$ by deleting the first row.*

Here, $p(n)$ denotes the number of partitions of $n$. In several places we shall make use of the well-known asymptotic formula

$$p(n) = \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)\left\{1 + \mathcal{O}(\frac{1}{\sqrt{n}})\right\}.$$

Apart from its intrinsic relevance, our interest in the behaviour of these multiplicities stems from a connection with the subgroup growth of a certain class of large groups, which we introduce next.

We work over the infinite alphabet $X = \{x_1, x_2, \ldots\}$. By a word we mean a finite string with symbols taken from $X \cup X^{-1}$. The empty word will be denoted by 1. We shall construct a collection of pairs $R = (w, S)$, where $w$ is a (typically large) word over $X$, and $S$ is a finite set of words of the form $x^q$ for some $x \in X$ and an integer $q \geq 2$. Denote by $X(R)$ the set of variables occurring in $R$. Let $\mathcal{R}$ be the smallest set of such pairs with the following properties.

(i) For each $i$ and every integer $e \geq 1$, we have $(x_i^e, \emptyset) \in \mathcal{R}$.
(ii) For each $i$ and every $q \geq 2$, we have $(1, x_i^q) \in \mathcal{R}$ and $(x_i, x_i^q) \in \mathcal{R}$.
(iii) If $(w_1, S_1), (w_2, S_2) \in \mathcal{R}$ with $X(w_1, S_1)$ and $X(w_2, S_2)$ disjoint, then $(w_1 w_2, S_1 \cup S_2) \in \mathcal{R}$.
(iv) If $(w, S) \in \mathcal{R}$ and $x \in X \setminus X(R)$, then $([w, x], S) \in \mathcal{R}$.

Moreover, we define functions $\mu, \alpha : \mathcal{R} \to \mathbb{Q} \cup \{\infty\}$ by recursion over $\mathcal{R}$ as follows.

(i) $\mu(x_i^e, \emptyset) = 1$, $\alpha(x_i, \emptyset) = \infty$, and $\alpha(x_i^e, \emptyset) = \frac{2}{e} - 1$ for $e \geq 2$.
(ii) $\mu(1, x_i^q) = \mu(x_i, x_i^q) = 1 - \frac{1}{q}$, $\alpha(x_i, x_i^q) = -\frac{1}{q}$, and $\alpha(1, x_i^q) = -1$.
(iii) If $(w_1, S_1), (w_2, S_2) \in \mathcal{R}$ with $X(w_1, S_1)$ and $X(w_2, S_2)$ disjoint, then $\mu(w_1 w_2, S_1 \cup S_2) = \mu(w_1, S_1) + \mu(w_2, S_2)$ and $\alpha(w_1 w_2, S_1 \cup S_2) = \alpha(w_1, S_1) + \alpha(w_2, S_2) + 1$.
(iv) If $(w, S) \in \mathcal{R}$ and $x \in X$ does not occur in $(w, S)$, then $\mu([w, x], S) = \mu(w, S) + 1$ and $\alpha([w, x], S) = \min(\alpha(w, S), 1)$.

Finally, denote by $d(w, S)$ the number of variables occurring in $w$, but not in $S$, and by $Q(w, S)$ the multiset of exponents occurring in $S$. For $R \in \mathcal{R}$, let $\Gamma_R = \langle X(R) | R \rangle$. Theorem 1 in conjunction with estimates from [5] allows us to establish the following.

**Theorem 2.** *Let $R = (w, S) \in \mathcal{R}$ with $w \neq 1$. If $\alpha(R) > 1$, we have*

$$s_n(\Gamma_R) \sim \delta_R\, n \cdot n!^{d(R)-2} \prod_{q \in Q(R)} |\operatorname{Hom}(C_q, S_n)| \tag{1}$$

*and*

$$s_n(\Gamma_{R_d}) \sim 2\, d\, n\, p(n)\, n!^{d(R)+d-2} \prod_{q \in Q(R)} |\operatorname{Hom}(C_q, S_n)|, \tag{2}$$

*where $R_d = ([w, x_1, \ldots, x_d], S)$ with variables $x_1, \ldots, x_d \notin X(R)$. Here, $\delta_R = 2$ if all words in $R$ are trivial on $C_2$, and 1 otherwise. In particular, two groups $\Gamma_{R_d}$ and $\Gamma_{R'_{d'}}$ with $\alpha(R), \alpha(R') > 1$ satisfy $s_n(\Gamma_{R_d}) \sim s_n(\Gamma_{R'_{d'}})$ if and only if $d(R) = d(R')$, $Q(R) = Q(R')$, $\delta_R = \delta_{R'}$, and $d = d'$. If $\alpha(R) \leq 1$, we still have*

$$s_n(\Gamma_R) < n!^{\mu(R)-2+(1-\alpha(R))/2+\varepsilon}.$$

Note that the asymptotic behaviour of the function $|\operatorname{Hom}(G, S_n)|$ with $G$ a finite group is well understood; cf. [3]. Hence, the right-hand sides of the asymptotic formulae given in Theorem 2 can be expressed in terms of elementary functions.

The class $\mathcal{R}$ is quite rich. First, it contains all sets of Fuchsian relations, in which case $\mu(R)-1$ is the hyperbolic measure and $\alpha(R)-1$ is the complexity measure introduced in [5]. Second, it contains all relations in class $\mathcal{W}_2$ as introduced in [4]. Moreover, it contains pairs $R$ of 'mixed' type, such as

$$R = ([[x_1x_2^5, x_3]x_4, x_5][x_6x_7^6, x_8], \{x_1^2, x_4^3, x_6^7\}).$$

On the other hand, words $w$ like $w = [x^a, y^b]$ are not allowed. Hence, Theorem 2 is a far reaching generalization of the asymptotic formula given by the main term in [5, Theorem A]. In principle, the asymptotic formula (1) can always be extended to an asymptotic expansion for some suitable sequence of gauge functions, since in this case the behaviour of $|\operatorname{Hom}(\Gamma_R, S_n)|$ is dominated by characters of small degree; in concrete examples of this type, these gauge functions have always turned out to be negative powers of $n$, but we have not attempted to prove this in general. Section 4 provides an example in this direction. In the case of formula (2) the situation is more complex due to the fact that all characters contribute terms of comparable magnitude.

Information on multiplicities can always be interpreted as a statement concerning distributions. By arguments similar to those employed in the proof of Theorem 1, we can show the following.

**Theorem 3.** *Let $f : S_n \to \mathbb{C}$ be a class function, and define $[f, x] : S_n \to \mathbb{C}$ by means of the twisted convolution*

$$[f, x](g) = \frac{1}{n!} \sum_{\substack{y,z \\ [y,z]=g}} f(y).$$

*Then there exists an absolute constant $C$ such that $\|[f, x]\|_2 \leq C\|f\|_2$.*

Here $\|f\|_2$ denotes the $L^2$ norm, defined for every function $f : S_n \to \mathbb{C}$ by means of the equation $\|f\|_2^2 = \frac{1}{n!}\sum_{g\in S_n} |f(g)|^2$.

## 2. Proofs of Theorems 1 and 3

We begin by collecting together a number of auxiliary results.

**Lemma 1.** *Let $n$ be sufficiently large, $\mathbf{c} \subseteq S_n$ a non-trivial conjugacy class, $s = s(\mathbf{c})$ the number of points moved by $\mathbf{c}$, and let $\lambda \vdash n$ be a partition.*

(i) *We have*

$$|\chi_\lambda(\mathbf{c})| \leq \chi_\lambda(1)^{1-\frac{\log(n/(n-s))}{14\log n}}$$

*if $s < n$, while, for fixed-point-free $\mathbf{c}$,*

$$|\chi_\lambda(\mathbf{c})| \leq \chi_\lambda(1)^{13/14}.$$

(ii) *We have*

$$|\chi_\lambda(\mathbf{c})| \leq \Big(\max\Big\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}, q\Big\}\Big)^{bs}\chi_\lambda(1)$$

*with certain absolute constants* $b > 0$ *and* $0 < q < 1$.

*Proof.* Part (i) is [5, Prop. 1], while part (ii) is the main result of [6]. □

**Lemma 2.** *Let* $\lambda \vdash n$ *be a partition, put* $n - \lambda_1 = k$*, and let* $\chi$ *be an arbitrary irreducible character of* $S_n$*. Then we have*

$$\langle \chi^2, \chi_\lambda\rangle \leq n^{k/2}\, k!.$$

*Proof.* By the branching rule[1] we have $\langle \chi_\lambda, \mathbf{1}\downarrow_{S_{n-k}}\uparrow^{S_n}\rangle > 0$, and therefore, using Frobenius reciprocity and orthogonality,

$$\begin{aligned}
\langle \chi^2, \chi_\lambda\rangle &\leq \langle \chi^2, \mathbf{1}\downarrow_{S_{n-k}}\uparrow^{S_n}\rangle \\
&= \langle \chi^2\downarrow_{S_{n-k}}, \mathbf{1}\rangle \\
&= \sum_{\psi\in \mathrm{Irr}(S_{n-k})} \langle \chi, \psi\rangle^2 \\
&\leq \max_{\psi\in \mathrm{Irr}(S_{n-k})} \langle \chi, \psi\rangle \sum_{\psi\in \mathrm{Irr}(S_{n-k})} \langle \chi, \psi\rangle.
\end{aligned}$$

The second factor equals the number of ways to remove $k$ boxes from the partition corresponding to $\chi$; this can be done in $\leq n^{k/2}$ ways, since a partition of $n$ has at most $\sqrt{n}$ exposed boxes. Also, a given partition $\mu \vdash n-k$ can be reached in $\leq k!$ ways, since, once the boxes to be removed have been chosen, the only thing remaining to be determined is the order in which they are removed. □

Although the statement of Lemma 2 holds for arbitrary $\lambda$, the estimate it provides is useful only if $k$ is sufficiently small, say $k < n^{1/4-\varepsilon}$. For larger $k$, we use the following.

**Lemma 3.** *Let* $\chi, \chi' \in \mathrm{Irr}(S_n)$ *be irreducible characters, and assume that* $\chi(1) \leq n!^{1/2-\delta}$ *with some* $\delta > 0$*. Then we have*

$$\langle \chi^2, \chi'\rangle \leq \chi'(1)\, n!^{-\delta} + \chi'(1)^{1-\frac{c}{\log n}}$$

*with some* $c > 0$ *which may depend on* $\delta$.

*Proof.* For a parameter $S$, we have

$$\begin{aligned}
\langle \chi^2, \chi'\rangle &\leq \chi(1)^2\,\chi'(1)\,\frac{\#\{\pi\in S_n : s(\pi)\leq S\}}{n!} + \max_{s(\pi)\leq S} |\chi'(\pi)| \\
&\leq \chi'(1)\,\frac{n^S}{n!^{2\delta}} + \chi'(1)^{1-\frac{\log n}{14\log(n/(n-S))}}.
\end{aligned}$$

Here we have used Lemma 1 (i). Choosing $S = \delta n/2$, our claim follows. □

[1]Cf., for instance, [1, Theorem 2.4.3].

**Lemma 4.** *Let $\lambda \vdash n$ be a partition, $\mu = \lambda \setminus \lambda_1$, and let $\pi \in S_n$ be a permutation. Then*

$$\chi_\lambda(\pi) = \sum_{\substack{\tilde{\mu}\subseteq\mu\\ \tilde{\mu}_1=1}} (-1)^{|\tilde{\mu}|} \sum_{\mathbf{c}\subseteq S_{|\mu|-|\tilde{\mu}|}} \chi_{\mu,\tilde{\mu}}(\mathbf{c}) \prod_{i\leq|\mu|} \binom{s_i(\pi)}{s_i(\mathbf{c})}, \tag{3}$$

*where $\mathbf{c}$ runs over all conjugacy classes of $S_{|\mu|-|\tilde{\mu}|}$, $\chi_{\mu,\tilde{\mu}}(\mathbf{c})$ denotes the number of ways to obtain $\tilde{\mu}$ from $\mu$ by removing rim hooks according to the cycle structure of $\mathbf{c}$, counted with the sign prescribed by the Murnaghan-Nakayama rule, and $s_i(\pi)$ is the number of $i$-cycles of $\pi$.*

*Proof.* This is [5, Lemma 7]. □

*Proof of Theorem* 1. The first statement is [7, Theorem 2.1], while the second follows from [7, Theorem 2.2]. To prove the third and fourth statement, write

$$\begin{aligned} m_\lambda &= \sum_{\mathbf{c}} \chi_\lambda(\mathbf{c}) \\ &= \chi_\lambda(1) \sum_{S=0}^{n} \sum_{s(\mathbf{c})=S} \frac{\chi_\lambda(\mathbf{c})}{\chi_\lambda(1)} \\ &\leq \chi_\lambda(1) \sum_{S=0}^{n} p(S) \max_{s(\mathbf{c})=S} \frac{\chi_\lambda(\mathbf{c})}{\chi_\lambda(1)} \\ &\leq \chi_\lambda(1) \sum_{S=0}^{n} p(S)\, e^{-c\frac{(n-\lambda_1)S}{n}}, \end{aligned}$$

where we estimated the character values using Lemma 1 (ii). If $n - \lambda_1 < c_1\sqrt{n}$, the largest summand is within the range $S \in [c'n, n]$ for some $c' > 0$ depending on $c_1$, and the fourth claim follows, whereas for larger values of $n - \lambda_1$ the largest value occurs in the range $S \asymp \frac{n^2}{(n-\lambda_1)^2}$, which implies the third claim.

To prove the fifth statement, note that, by Lemma 2, we have

$$\sum_{\chi} \langle \chi^2, \chi_\lambda \rangle \leq n^{k/2}\, k!\, p(n),$$

where $k = n - \lambda_1$. On the other hand, by [5, Lemma 8 (ii)], $\chi_\lambda(1) \geq \binom{n-k}{k}$, thus

$$\frac{m_{\chi_\lambda}}{p(n)\chi_\lambda(1)} \leq \frac{n^{k/2}(k!)^2}{(n-k)^k} \ll \frac{(k!)^2}{n^{k/2}} < \left(\frac{k^4}{n}\right)^{k/2}.$$

For the last statement we compute $\sum \chi(\mathbf{c})$ using Lemma 4. We first claim that the only relevant terms come from conjugacy classes $\mathbf{c}$ with $s_i(\mathbf{c}) < n^{1/2+\varepsilon}$ for all $i$. In fact, the number of partitions with at least $\ell$ parts of length $i$ is

$$p(n - \ell i) \ll p(n)\, e^{-c\frac{\ell i}{\sqrt{n}}};$$

hence, the number of partitions with some part repeated $n^{1/2+\varepsilon}$ times is bounded above by $p(n)e^{-n^\varepsilon}$, and the sum over such conjugacy classes is at most $p(n)\chi_\lambda(1)e^{-n^\varepsilon} < \frac{p(n)}{n}$.

With $k$ as above, we see that in (3) all terms of degree $\leq k-2$ are smaller then the main term by a factor $n^{1-2\varepsilon}$. Thus, we obtain

$$m_\lambda = \chi_\mu(1) \sum_{s_i(\mathbf{c})<n^{1/2+\varepsilon}} \Big(\frac{s_1(\mathbf{c})^k}{k!} + \frac{s_1(\mathbf{c})^{k-2}s_2(\mathbf{c})}{(k-2)!} + \frac{s_1(\mathbf{c})^{k-1}}{(k-1)!}\Big) + \mathcal{O}(n^{k/2-1+2\varepsilon}p(n)).$$

Now we can re-introduce conjugacy classes with more then $n^{1/2+\varepsilon}$ $i$-cycles without changing the error term, and compute

$$\sum_{\mathbf{c}} s_1(\mathbf{c})^k = \sum_{\nu\geq 1} \nu^k\big(p(n-\nu) - p(n-\nu-1)\big) = \sum_{\nu\geq 1}(\nu^k - (\nu-1)^k)\, p(n-\nu).$$

Summands with $\nu > n/2$ are negligible, while for smaller values of $\nu$ we may replace $p(n)$ by its asymptotic formula to obtain

$$\big(1+\mathcal{O}(n^{-1/2})\big) \sum_{\nu\geq 1}(\nu^k - (\nu-1)^k)\, \frac{e^{\frac{2\pi\sqrt{n-\nu}}{\sqrt{6}}}}{4(n-\nu)\sqrt{3}}.$$

Approximating the series by an integral, we obtain

$$\sum_{\mathbf{c}} s_1(\mathbf{c})^k = \big(1+\mathcal{O}(n^{-1/2})\big)\, k!\, \Big(\frac{\sqrt{6}}{\pi}\Big)^k n^{k/2}.$$

In the same way we find that the second and third summand is smaller by a factor $n^{1/2}$, and we obtain

$$m_\lambda = \big(1+\mathcal{O}(n^{-1/2})\big)\, k!\, \Big(\frac{\sqrt{6}}{\pi}\Big)^k \chi_\mu(1)\, n^{k/2},$$

as claimed. □

*Proof of Theorem* 3. Write $f = \sum_\lambda \hat{f}(\lambda)\chi_\lambda$. Parseval's equation yields $\|f\|_2^2 = \sum_\lambda \hat{f}(\lambda)^2$. Hence, we obtain

$$\begin{aligned}
\|[f,x]\|_2^2 &= \sum_\lambda \widehat{[f,x]}(\lambda)^2 \\
&= \sum_\lambda \frac{1}{\chi_\lambda(1)^2}\Big(\sum_\mu \hat{f}(\mu)\langle\chi_\lambda^2,\chi_\mu\rangle\Big)^2 \\
&\leq \sum_\lambda \frac{1}{\chi_\lambda(1)^2}\Big(\sum_\mu \hat{f}(\mu)^2\Big)\Big(\sum_\mu \langle\chi_\lambda^2,\chi_\mu\rangle^2\Big) \\
&= \|f\|_2^2 \sum_{\lambda,\mu} \frac{\langle\chi_\lambda^2,\chi_\mu\rangle^2}{\chi_\lambda(1)^2} \\
&= \|f\|_2^2 \sum_\lambda \frac{\langle\chi_\lambda^2,\chi_\lambda^2\rangle}{\chi_\lambda(1)^2}.
\end{aligned}$$

Thus, it suffices to show that the last character sum can be bounded independently of $n$. Consider first the sum extended over partitions $\lambda$ with $\lambda_1 \leq n/2$, where we may assume $\lambda_1 \geq \lambda_1'$. We have

$$\begin{aligned}\sum_{\lambda_1\leq n/2} \frac{\langle \chi_\lambda^2, \chi_\lambda^2\rangle}{\chi_\lambda(1)^2} &= \frac{1}{n!}\sum_S \sum_{\lambda_1\leq n/2} \sum_{s(\pi)=S} \frac{\chi_\lambda(\pi)^4}{\chi_\lambda(1)^2}\\ &\leq \frac{1}{n!}\sum_S \sum_{s(\pi)=S} \left(\max_{\lambda_1\leq n/2} \frac{\chi_\lambda(\pi)^2}{\chi_\lambda(1)^2}\right) \sum_{\lambda\vdash n} \chi_\lambda(\pi)^2\\ &= \sum_S \sum_{s(\mathbf{c})=S} \max_{\lambda_1\leq n/2} \frac{\chi_\lambda(\mathbf{c})^2}{\chi_\lambda(1)^2}\\ &\leq \sum_S e^{c\sqrt{S}} \max_{s(\mathbf{c})=S} \max_{\lambda_1\leq n/2} \frac{\chi_\lambda(\mathbf{c})^2}{\chi_\lambda(1)^2}\\ &\leq \sum_S e^{c\sqrt{S}-c'S}\\ &\ll 1.\end{aligned}$$

If on the other hand $\lambda_1 > n/2$, then, splitting the range of summation for $\langle \chi_\lambda^2, \chi_\lambda^2\rangle$ into permutations $\pi$ with $s(\pi) \leq S$ and $s(\pi) > S$, respectively, we obtain for given $S$ that

$$\frac{\langle \chi_\lambda^2, \chi_\lambda^2\rangle}{\chi_\lambda(1)^2} \leq \frac{1}{n!}\chi_\lambda(1)^2 \binom{n}{S} S! + \left(\frac{\lambda_1}{n}\right)^{-cS}.$$

Since $\lambda_1 > n/2$ implies $\chi_\lambda(1) < n!^{1/3}$, the first summand is negligible, provided that $S \leq n/4$. Summing over all $\lambda$, the second summand results in a contribution, which is of order

$$\sum_{k\geq 0} e^{c\sqrt{k}}\left(\frac{n-k}{n}\right)^{c'n} \leq \sum_{k\geq 0} e^{c\sqrt{k}-c'k} \ll 1.$$

□

**Remark.** Note that building random commutators may concentrate a distribution as well as flatten it. For example, if $f$ is equidistributed on $S_n$, $\|[f,x]\|_2 > 2\|f\|_2$, while if $f$ is equidistributed on transpositions and vanishes elsewhere, then $\|[f,x]\|_2 \ll \|f\|_2 n^{-2}$.

## 3. Proof of Theorem 2

In view of the transformation formula [2, Prop. 1], it suffices to obtain an asymptotic formula for $|\operatorname{Hom}(\Gamma_R, S_n)|$. For $R = (w, S) \in \mathcal{R}$, define $N_R(\pi)$ to be the number of solutions of the equation $w(\vec{x}) = \pi$ with tuples $\vec{x}$ satisfying $S$. Since $N_R(\pi)$ is a class function, so is

$$F_R(\pi) = \frac{n! N_R(\pi)}{\sum_\sigma N_r(\sigma)},$$

and we can define Fourier coefficients $\hat{F}_R(\lambda)$. We claim that $|\hat{F}_R(\lambda)| \leq \chi_\lambda(1)^{-\alpha(R)+\varepsilon}$, and $\sum_\sigma N(\sigma) = (n!)^{\mu(R)+o(1)}$. From these bounds the first assertion of the theorem is obtained as follows. We have

$$|\operatorname{Hom}(\Gamma_R, S_n)| = N_R(1) = \sum_\lambda \hat{F}_R(\lambda)\,\chi_\lambda(1).$$

The contribution of the trivial character is

$$\frac{1}{n!}\sum_{\sigma} N_R(\sigma) = n!^{d(R)-1}\prod_{q\in Q(R)} |\operatorname{Hom}(C_q, S_n)|,$$

while the contribution of the sign character is the same if $R$ is trivial on $C_2$, and 0 if $R$ is not trivial on $C_2$. Hence, it suffices to show that the contribution of non-linear characters is of lesser order. This quantity is

$$\Big|\sum_{\lambda\neq(n),(1^n)} \hat{F}_R(\lambda)\,\chi_\lambda(1)\Big| \leq \sum_{\lambda\neq(n),(1^n)} \chi_\lambda(1)^{1-\alpha_R+\varepsilon} \ll n^{1-\alpha_R+\varepsilon},$$

which is sufficiently small for $\alpha(R) > 1$. If $\alpha(R)\leq 1$, the linear characters no longer dominate the sum, and we obtain

$$N_R(1) \leq (n!)^{\mu(R)}\sum_{\lambda} \chi_\lambda(1)^{1-\alpha_R+\varepsilon} \ll (n!)^{\mu(R)+(1-\alpha(R))/2+\varepsilon}.$$

Since $s_n(\Gamma_R) \leq \frac{|\operatorname{Hom}(\Gamma_R, S_n)|}{(n-1)!}$, this proves the last assertion. We postpone the proof of the remaining assertion until we have established our claim concerning the Fourier coefficients $\hat{F}_R(\lambda)$.

The estimate for $\hat{F}_R(\lambda)$ is proved by induction on the construction of $R$. If $R = (x_i, \emptyset)$, then $N_R(\pi) = 1$ for all $\pi$ while $\hat{F}_R(\lambda) = 0$ for all partitions $\lambda \neq (n)$. If $R = (x_i^e, \emptyset)$ with $e \geq 2$, our claim follows from [5, Prop. 2], whereas for $R = (x_i, x_i^q)$ it is [5, Prop. 1]. If $R = (1, x_i^q)$, the function $N_R(\pi)$ yields the character of the regular representation, and therefore $\hat{F}_R(\lambda) = \chi_\lambda(1)$, while $\alpha(R) = -1$. If $R_1 = (w_1, S_1)$, $R_2 = (w_2, S_2)$ with $X(R_1)$ and $X(R_2)$ disjoint, and $R = (w_1w_2, S_1\cup S_2)$, then $\hat{F}_R(\lambda) = \frac{\hat{F}_{R_1}(\lambda)\hat{F}_{R_2}(\lambda)}{\chi_\lambda(1)}$; cf., for instance, [4, Lemma 1 (ii)]. If $R = (w, S)$, $x \in X \setminus X(R)$, and $R' = ([w, x], S)$, then, by [4, Lemma 1 (iii)],

$$\hat{F}_{R'}(\lambda) = \frac{1}{\chi_\lambda(1)}\sum_{\mu} \hat{F}_R(\mu)\langle\chi_\lambda^2, \chi_\mu\rangle. \tag{4}$$

We have

$$\langle\chi_\lambda^2, \chi_\mu\rangle \leq \frac{1}{n!}\sum_{\pi\in S_n} \chi_\lambda(\pi)^2|\chi_\mu(\pi)| \leq \frac{\chi_\mu(1)}{n!}\sum_{\pi\in S_n}\chi_\lambda(\pi)^2 = \chi_\mu(1)$$

as well as

$$\chi_\lambda(1)^2 = \sum_{\nu\vdash n}\langle\chi_\lambda^2, \chi_\nu\rangle\chi_\nu(1) \geq \langle\chi_\lambda^2, \chi_\mu\rangle\chi_\mu(1).$$

Combining these estimates, we obtain

$$\langle\chi_\lambda^2, \chi_\mu\rangle \leq \min\Big(\chi_\mu(1), \frac{\chi_\lambda(1)^2}{\chi_\mu(1)}\Big).$$

Inserting this estimate into (4), we obtain

$$\begin{aligned}\hat{F}_{R'}(\lambda) &\leq \frac{1}{\chi_\lambda(1)}\sum_{\langle\chi_\lambda^2,\chi_\mu\rangle>0} \chi_\mu(1)^{-\alpha(R)}\min\Big(\chi_\mu(1), \frac{\chi_\lambda(1)^2}{\chi_\mu(1)}\Big)\\ &\leq \chi_\lambda(1)^{-\alpha(R)}\#\big\{\mu\vdash n : \chi_\mu(1)\leq\chi_\lambda(1)^2\big\}.\end{aligned}$$

We claim that the second factor is $\leq \chi_\lambda(1)^\varepsilon$ for all $\lambda$, which would imply the desired bound for $\hat{F}_{R'}(\lambda)$. This estimate is trivial, provided $\chi_\lambda(1) > e^{\sqrt{n}\log n}$ or $\chi_\lambda(1) = 1$. Otherwise, we may suppose that $\lambda_1 \geq \lambda_1'$, and, writing $k = n - \lambda_1 \leq \sqrt{n}\log n$, we estimate $\chi_\lambda(1)$ as

$$\binom{n-k}{k} \leq \chi_\lambda(1) \leq \binom{n}{k} k!^{1/2}.$$

Hence, every partition $\mu$ with $\chi_\mu(1) < \chi_\lambda(1)^2$ satisfies $n - \mu_1 < 3\sqrt{n}\log n$ as well as

$$\binom{\mu_1}{n-\mu_1} \leq \chi_\mu(1) \leq \chi_\lambda(1)^2 \leq \binom{n}{n-\lambda_1}^2 (n-\lambda_1)!.$$

These inequalities imply $n - \mu_1 \leq 4(n - \lambda_1)$; thus, the number of partitions $\mu$ with $\chi_\mu(1) \leq \chi_\lambda(1)^2$ is $\leq p(4d)$. Our claim now follows from the estimate

$$\binom{n-d}{d}^\varepsilon \geq p(4d),$$

valid for all $d$ in the range $1 \leq d \leq \frac{n}{3}$. We now prove the asymptotic formula for $s_n(\Gamma_{R_d})$. Iterating formula (4), we find that

$$\frac{N_{R_d}(1)n!}{\sum_\sigma N_{R_d}(\sigma)} = \sum_{\lambda^{(1)},\dots,\lambda^{(d+1)}} \frac{\langle \chi^2_{\lambda^{(1)}}, \chi_{\lambda^{(2)}}\rangle}{\chi_{\lambda^{(2)}}(1)} \frac{\langle \chi^2_{\lambda^{(2)}}, \chi_{\lambda^{(3)}}\rangle}{\chi_{\lambda^{(3)}}(1)} \cdots \frac{\langle \chi^2_{\lambda^{(d-1)}}, \chi_{\lambda^{(d)}}\rangle}{\chi_{\lambda^{(d)}}(1)} \langle \chi^2_{\lambda^{(d)}}, \chi_{\lambda^{(d+1)}}\rangle \hat{F}_R(\lambda^{(d+1)}).$$

We split the sum according to the size of $\lambda_1^{(2)}$. Suppose first that $\lambda_1^{(2)} = o(n)$. Then we apply Theorem 1 (i) and obtain that for each $\varepsilon > 0$ there exist $\delta > 0$ and $\theta$ with $|\theta| < 1$, such that

$$\sum_{\substack{\lambda^{(1)},\dots,\lambda^{(d)} \\ \lambda_1^{(2)} \leq \delta n}} \frac{\langle \chi^2_{\lambda^{(1)}}, \chi_{\lambda^{(2)}}\rangle}{\chi_{\lambda^{(2)}}(1)} \frac{\langle \chi^2_{\lambda^{(2)}}, \chi_{\lambda^{(3)}}\rangle}{\chi_{\lambda^{(3)}}(1)} \cdots \frac{\langle \chi^2_{\lambda^{(d-1)}}, \chi_{\lambda^{(d)}}\rangle}{\chi_{\lambda^{(d)}}(1)} \langle \chi^2_{\lambda^{(d)}}, \chi_{\lambda^{(d+1)}}\rangle \hat{F}_R(\lambda^{(d+1)})$$
$$= (1+\theta\varepsilon) \sum_{\substack{\lambda^{(2)},\dots,\lambda^{(d)} \\ \lambda_1^{(2)} \leq \delta n}} \frac{\langle \chi^2_{\lambda^{(2)}}, \chi_{\lambda^{(3)}}\rangle}{\chi_{\lambda^{(3)}}(1)} \cdots \frac{\langle \chi^2_{\lambda^{(d-1)}}, \chi_{\lambda^{(d)}}\rangle}{\chi_{\lambda^{(d)}}(1)} \langle \chi^2_{\lambda^{(d)}}, \chi_{\lambda^{(d+1)}}\rangle \hat{F}_R(\lambda^{(d+1)}).$$

If $\lambda^{(2)} = (n)$, that is, $\chi_{\lambda^{(2)}}$ is trivial, all summands with $\chi_{\lambda^{(i)}}$ non-trivial for some $i \geq 3$ vanish, while the other summands are all equal to 1. Hence, these terms contribute $p(n)$. We now show that the total contribution of the remaining cases is of lesser order of magnitude.

Suppose that $\lambda_1^{(2)} > \delta n$. Then $\chi_{\lambda^{(2)}}(1) < \frac{n!}{(n-\lambda_1^{(2)})!^{1/2}\lambda_1^{(2)}!}$, since the hook product extended over all boxes outside the first row is at least $(n-\lambda_1^{(2)})!^{1/2}$, while the product in the first row is at least $\lambda_1^{(2)}!$. From this, we deduce $\chi_\lambda(1) < n!^{1/2-\delta/2}$, and we can apply Lemma 3 to obtain $\langle \chi^2_{\lambda^{(2)}}, \chi_{\lambda^{(3)}}\rangle \leq \chi_{\lambda^{(3)}}(1)^{1-\frac{c}{\log n}}$. The number of summands in the whole sum is bounded above by $p(n)^d$, and each single term with $\chi_{\lambda^{(3)}}(1) > e^{\sqrt{n}\log^2 n}$ is less than

$e^{-c\sqrt{n}\log n}$; hence, these terms can be neglected. If furthermore $n-\lambda_1^{(2)} > \delta n$, we have $\sum_\chi \langle \chi^2, \chi_{\lambda^{(2)}}\rangle \ll 1$ by Theorem 1 (ii). In this situation, either one of the multiplicities $\langle \chi^2_{\lambda^{(i)}}, \chi_{\lambda^{(i+1)}}\rangle$ vanishes, or all partitions $\lambda^{(i)}$, $i\geq 4$, have $\mathcal{O}(n-\lambda_1^{(3)})$ boxes outside the first row. The number of such partitions is $e^{c(n-\lambda_1^{(3)})^{1/2}}$; hence, the sum extended over $\lambda^{(4)},\dots,\lambda^{(d)}$ is bounded above by

$$\chi_{\lambda^{(3)}}(1)^{-\frac{c}{\log n}} e^{c(n-\lambda_1^{(3)})^{1/2}} \leq \chi_{\lambda^{(3)}}(1)^{-\frac{c'}{\log n}},$$

and the sum extended over all $\lambda^{(3)}$ with $n-\lambda_1^{(3)} < \sqrt{n}\log n$ is bounded independently of $n$. Summarizing these estimates, we obtain

$$\begin{aligned} \sum_{\substack{\lambda^{(1)},\dots,\lambda^{(d+1)}\\ \lambda_1^{(2)}>\delta n}} & \frac{\langle \chi^2_{\lambda^{(1)}}, \chi_{\lambda^{(2)}}\rangle}{\chi_{\lambda^{(2)}}(1)} \frac{\langle \chi^2_{\lambda^{(2)}}, \chi_{\lambda^{(3)}}\rangle}{\chi_{\lambda^{(3)}}(1)} \cdots \frac{\langle \chi^2_{\lambda^{(d-1)}}, \chi_{\lambda^{(d)}}\rangle}{\chi_{\lambda^{(d)}}(1)} \langle \chi^2_{\lambda^{(d)}}, \chi_{\lambda^{(d+1)}}\rangle \hat{F}_R(\lambda^{(d+1)}) \\ \leq & \sum_{\substack{\lambda^{(1)},\dots,\lambda^{(d+1)}\\ \lambda_1^{(2)}>\delta n}} \frac{\langle \chi^2_{\lambda^{(1)}}, \chi_{\lambda^{(2)}}\rangle}{\chi_{\lambda^{(2)}}(1)} \frac{\langle \chi^2_{\lambda^{(2)}}, \chi_{\lambda^{(3)}}\rangle}{\chi_{\lambda^{(3)}}(1)} \cdots \frac{\langle \chi^2_{\lambda^{(d)}}, \chi_{\lambda^{(d+1)}}\rangle}{\chi_{\lambda^{(d+1)}}(1)^{\alpha(R)-\varepsilon}} \\ \ll & \sum_{\substack{\lambda^{(1)},\lambda^{(2)}\\ \lambda_1^{(2)}>\delta n}} \frac{\langle \chi^2_{\lambda^{(1)}}, \chi_{\lambda^{(2)}}\rangle}{\chi_{\lambda^{(2)}}(1)}. \end{aligned}$$

Note that the number of partitions $\lambda^{(2)}$ with $\lambda_1^{(2)} > \delta n$ is $\leq p(n)e^{-c\sqrt{n}}$; thus, the contribution of characters $\chi_{\lambda^{(2)}}$ satisfying

$$m_{\lambda^{(2)}} = \sum_\lambda \langle \chi_\lambda^2, \chi_{\lambda^{(2)}}\rangle \leq e^{c\sqrt{n}}\chi_{\lambda^{(2)}}(1)$$

is negligible, provided $c$ is sufficiently small. By Theorem 1 (iii), we may therefore assume $n-\lambda_1^{(2)} \leq C\sqrt{n}$, where $C$ has to be chosen sufficiently large. Thus, we may apply parts (iv) and (v) of Theorem 1 to obtain

$$\begin{aligned} \sum_{\substack{\lambda\\ 1\leq\lambda_1\leq C\sqrt{n}}} \frac{m_{\chi_\lambda}}{\chi_\lambda(1)} & \leq p(n) \sum_{\nu>n^{1/4}/2} e^{-c\nu} p(\nu) + p(n) \sum_{1\leq\nu\leq n^{1/4}/2} \left(\frac{\nu^4}{n}\right)^{\nu/2} p(\nu) \\ & \leq p(n)\, e^{-cn^{1/4}} + p(n)\, e^{-cn^{1/8}} + p(n) \sum_{1\leq\nu\leq n^{1/8}} n^{-\nu/4} e^{c\nu} \\ & \leq p(n)\, n^{-1/4}. \end{aligned}$$

Denoting the $\tau$-fold character sum with $S_\tau$, we have shown that

$$S_\tau = \big(1+o(1)\big)S_{\tau-1} + \big(1+\mathcal{O}(n^{-1/4})\big)p(n).$$

Finally, $S_2$ can be computed as

$$\sum_{\lambda^{(1)},\lambda^{(2)}} \langle \chi^2_{\lambda^{(1)}}, \chi_{\lambda^{(2)}}\rangle \hat{F}_R(\lambda^{(2)}).$$

Here, summands with $\chi_{\lambda^{(2)}}$ trivial contribute $p(n)$, summands with $\chi_{\lambda^{(2)}}(1) > e^{\sqrt{n}\log n}$ make a negligible contribution since $\alpha(R) > 1$, and summands with intermediate size of

$\chi_{\lambda^{(2)}}(1)$ can be estimated as above; hence, $S_2 \sim p(n)$, and we obtain $S_\tau \sim (\tau-1)p(n)$. Inserting this result into the definition of the Fourier coefficients, we obtain

$$|\operatorname{Hom}(\Gamma_{R_d}, S_n)| \sim \delta_R \, d \, p(n) \, n!^{d(R)+d-1} \prod_{q\in Q(R)} |\operatorname{Hom}(C_q, S_n)|,$$

from which the asymptotic formula for $s_n(\Gamma_{R_d})$ follows.

To prove that for groups $\Gamma_{R_d}$ with $\alpha(R) > 1$ the parameters $d(R), d$ and $Q(R)$ are uniquely determined by the asymptotics of $s_n(\Gamma_{R_d})$, suppose that $R, R' \in \mathcal{R}$ and $d, d' \in \mathbb{N}$ are given such that $s_n(\Gamma_{R_d}) \sim s_n(\Gamma_{R'_{d'}})$. Suppose first that $d, d' > 0$. Then we deduce

$$(d-1)n!^{d(R)+d-2} \prod_{q\in Q(R)} |\operatorname{Hom}(C_q, S_n)| \sim (d'-1)n!^{d(R')+d'-2} \prod_{q\in Q(R')} |\operatorname{Hom}(C_q, S_n)|. \quad (5)$$

As a first approximation, we may take $\log |\operatorname{Hom}(C_q, S_n)| \sim (1-\frac{1}{q}) \log n!$, which implies

$$d(R) + d + \sum_{q\in Q(R)} 1 - \frac{1}{q} = d(R') + d' + \sum_{q\in Q(R')} 1 - \frac{1}{q}.$$

The more precise result

$$|\operatorname{Hom}(C_q, S_n)| \sim K_q n!^{1-1/q} (2\pi n)^{-1+1/q} \exp\Big(\sum_{\substack{d|q\\ d<q}} n^{d/q}/d\Big),$$

where $K_q = q^{-1/2}$, if $q$ is odd, and $K_q = q^{-1/2}e^{-1/(2q)}$, if $q$ is even, which is a special case of [3], allows to write (5) as

$$d\Big(\prod_{q\in Q(R)} K_q (2\pi n)^{-1+1/q}\Big) \exp\Big(\sum_{q\in Q(R)} \sum_{\substack{d|q\\ d<q}} n^{d/q}/d\Big)$$
$$\sim d'\Big(\prod_{q\in Q(R')} K_q (2\pi n)^{-1+1/q}\Big) \exp\Big(\sum_{q\in Q(R')} \sum_{\substack{d|q\\ d<q}} n^{d/q}/d\Big). \quad (6)$$

Considering only factors of super-polynomial order, we obtain for each integer $t \geq 2$ the equation

$$\sum_{\substack{q\in Q(R)\\ t|q}} \frac{t}{q} = \sum_{\substack{q\in Q(R')\\ t|q}} \frac{t}{q},$$

hence, $Q(R) = Q(R')$ as multisets. Now (5) becomes $(d-1) \sim (d'-1)$, and our claim follows.

The same argument applies, if both $d$ and $d'$ vanish. Finally, if $d = 0$ and $d' > 0$, the same computation as above together with the asymptotic formula for $p(n)$ yields instead of (6) the relation

$$\delta_R\Big(\prod_{q\in Q(R)} K_q(2\pi n)^{-1+1/q}\Big)\exp\Big(\sum_{q\in Q(R)}\sum_{\substack{d\mid q\\ d<q}} n^{d/q}/d\Big)$$

$$\sim \frac{d'}{4n\sqrt{3}}\Big(\prod_{q\in Q(R')} K_q(2\pi n)^{-1+1/q}\Big)\exp\Big(\pi\sqrt{\frac{2n}{3}}+\sum_{q\in Q(R')}\sum_{\substack{d\mid q\\ d<q}} n^{d/q}/d\Big).$$

Taking logarithms and considering only terms of highest order, we deduce

$$\sum_{\substack{q\in Q(R)\\ 2\mid q}}\frac{\sqrt{n}}{q/2} = \pi\sqrt{\frac{2n}{3}} + \sum_{\substack{q\in Q(R')\\ 2\mid q}}\frac{\sqrt{n}}{q/2} + \mathcal{O}(n^{1/3}),$$

which is absurd, since $\pi$ is transcendent.

## 4. An Example

As can be seen from the proof of Theorem 2, restricting the formula for $|\operatorname{Hom}(\Gamma_R, S_n)|$ to partitions $\lambda$ with $n-\lambda_1 \leq K$ or $n-\lambda_1' \leq K$ introduces an error term of size

$$\sum_{\substack{\lambda\\ \lambda_1<n-c,\ \lambda_1'<n-c}} \hat{F}_R(\lambda)\,\chi_\lambda(1) \ll n^{-K(\alpha(R)-1)}.$$

Hence, taking more and more characters into account, one obtains expressions for $|\operatorname{Hom}(\Gamma_R, S_n)|$ with relative error of order $\mathcal{O}(n^{-A})$ for arbitrarily large $A$. Hence, one would expect the existence of an asymptotic series for $s_n(\Gamma_R)$ with main term as in (1) and gauge functions expressible in terms of some sort of elementary functions; however, due to the possibly highly involved nature of $R$ and the possibility, that random commutators might mix better than expected, it is not clear in general which functions of $n$ would occur as gauge functions in such a series. Still, for any given $R$ these difficulties may be overcome. Here, we shall consider the set of relations

$$R = ([x_1^a, x_2][x_3, x_4][x_5, x_6, x_7][x_8, x_9], \{x_3^q\}), \quad a\geq 2,\ q\geq 2,$$

which is sufficiently complex to indicate all phenomena to be expected in general. We see that

$$\alpha(R) = \frac{2}{a} - \frac{1}{q} + 4 > 1;$$

hence, the sum over characters yields an asymptotic formula for $s_n(\Gamma_R)$. To compute the contribution of the fixed point character to $|\operatorname{Hom}(\Gamma_R, S_n)|$, we need to evaluate $\hat{F}_R((n-1,1))$. We have

$$\hat{F}_{[x_1^a,x_2]}((n-1,1)) = \frac{1}{n-1}\Big(\hat{F}_{x_1^a}((n)) + \hat{F}_{x_1^a}((n-1,1))\langle\chi^2_{(n-1,1)}, \chi_{(n-1,1)}\rangle + \hat{F}_{x_1^a}((n-2,1,1))\langle\chi^2_{(n-1,1)}, \chi_{(n-2,1,1)}\rangle + \hat{F}_{x_1^a}((n-2,2))\langle\chi^2_{(n-1,1)}, \chi_{(n-2,2)}\rangle\Big).$$

The occurring multiplicities $\langle \chi_\lambda^2, \chi_\mu\rangle$ are easily computed[2] (in fact, they all equal 1), while the coefficients $\hat{F}_{x_1^a}(\lambda)$ for partitions $\lambda$ with $n-\lambda_1 \leq 2$ have been computed in [5, Prop. 2 (ii)]. Inserting these values into the last equation, we obtain

$$\hat{F}_{[x_1^a, x_2]}((n-1,1)) = \frac{1}{n-1}\Big(\sigma(a) + \tau(a)^2 - 2\tau(a) + 1\Big),$$

where $\tau(a)$ denotes the number of divisors of $a$, and $\sigma(a)$ is the sum of divisors of $a$. Moreover, in general the multiplicities $\langle \chi_\lambda^2, \chi_\mu\rangle$ are always bounded, provided that $n-\lambda_1, n-\mu_1$ are bounded, and the same holds for the Fourier coefficients $\hat{F}_{x_1^a}(\lambda)$; hence, for $n-\lambda$ fixed, we have $\hat{F}_{[x_1^a,x_2]}(\lambda) \ll \frac{1}{\chi_\lambda(1)}$. In particular, if $R$ is of the form $([x_1^{a_1}, y_1]\dots[x_d^{a_d}, y_d], \emptyset)$, one obtains an asymptotic formula for $s_n(\Gamma_R)$ in terms of the functions $n^{-k}$, $k\in\mathbb{N}$.

To compute $\hat{F}_{([x,y],\{x^q\})}((n-1,1))$, we proceed as above, this time referring to [5, Lemma 21], and obtain

$$\begin{aligned}
&\hat{F}_{([x,y],\{x^q\})}((n-1,1))\\
&\quad= \frac{1}{n-1}\Big(1 + \hat{F}_{(x,\{x^q\})}((n-1,1)) + \hat{F}_{(x,\{x^q\})}((n-2,1,1)) + \hat{F}_{(x,\{x^q\})}((n-2,2))\Big)\\
&\quad= \frac{1}{n-1}\Big(1 + n^{1/q} + n^{2/q} + \mathcal{O}(n^{\frac{2}{q}-\frac{1}{2}})\Big).
\end{aligned}$$

Moreover, for $n-\lambda_1$ fixed, we have $\hat{F}_{([x,y],\{x^q\})}(\lambda) \ll \chi_\lambda(1)^{-1+\frac{2}{q}}$.

Next, we consider $\hat{F}_{[x,y,z]}((n-1,1))$. We have

$$\hat{F}_{[x,y,z]}((n-1,1)) = \frac{1}{n-1}\sum_\lambda \frac{\langle \chi_{(n-1,1)}^2, \chi_\lambda\rangle}{\chi_\lambda(1)} = \frac{n^4 - 5n^3 + 10n^2 - 12n + 4}{(n-1)^2(n-2)n(n-3)};$$

and, for fixed $n-\lambda_1$, we have the bound $\hat{F}_{[x,y,z]}(\lambda) \ll \frac{1}{\chi_\lambda(1)}$. Finally, $\hat{F}_{[x,y]}(\lambda) = \frac{1}{\chi_\lambda(1)}$. Putting together all these estimates, we deduce

$$\hat{F}_R((n-1,1)) = \big(1 + \mathcal{O}(n^{-1/2})\big)\frac{(1+n^{1/q}+n^{2/q})(\sigma(a)+\tau(a)^2-2\tau(a)+1)}{n^7},$$

and, for $n-\lambda_1 \geq 2$ fixed, $\hat{F}_R(\lambda) \ll \chi_\lambda(1)^{-7+\frac{2}{q}} \ll n^{-8}$. Note that $\delta_R = 2$ if $q$ is even, and $\delta_R = 1$ otherwise. Putting these results into the explicit formula for $|\operatorname{Hom}(\Gamma_R, S_n)|$, we find that

$$\begin{aligned}
|\operatorname{Hom}(\Gamma_R, S_n)| &= \delta_R\, n!^8\, |\operatorname{Hom}(C_q, S_n)|\\
&\qquad\times\Big(1 + \frac{(1+n^{1/q}+n^{2/q})(\sigma(a)+\tau(a)^2-2\tau(a)+1)}{n^6} + \mathcal{O}(n^{-13/2+\frac{2}{q}})\Big).
\end{aligned}$$

[2]The general formalism for dealing with this type of problem can be found in [1, Sec. 2.9]; our situation is simple enough to allow for a more elementary approach using only character degrees.

Finally, due to the rapid growth of $|\operatorname{Hom}(\Gamma_R, S_n)|$, the number of non-transitive $\Gamma_R$-operations on $\{1,\dots,n\}$ is $\ll n^{-8+\frac{1}{q}}|\operatorname{Hom}(\Gamma_R, S_n)|$, as can be seen from the transformation formula [2, Prop. 1], and we conclude that

$$s_n(\Gamma_R) = \delta_R\, n\, n!^7\, |\operatorname{Hom}(C_q, S_n)| \times \Big(1 + \frac{(1+n^{1/q}+n^{2/q})(\sigma(a)+\tau(a)^2-2\tau(a)+1)}{n^6} + \mathcal{O}(n^{-13/2+\frac{2}{q}})\Big).$$

In general, computation even of the first term of the asymptotic series becomes extremely tedious. If $R$ involves a $k$-fold commutator, all Fourier coefficients corresponding to partitions $\lambda$ with $n-\lambda_1 \leq 2^{k-1}$ of a certain subword may influence $\hat{F}_R((n-1,1))$; hence, in order to compute the simplest non-trivial Fourier coefficient of the final word, one has to consider $p(2^{k-1})$ individual terms.

## References


[1] G. James, A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of mathematics and its applications vol. 16, Cambridge University Press, 1984.

[2] T. Müller, Combinatorial aspects of finitely generated virtually free groups, *J. London Math. Society* **44**, 75–94.

[3] T. W. Müller, Finite group actions and asymptotic expansion of $e^{P(z)}$, *Combinatorica* **17** (1997), 523–554.

[4] T. W. Müller, J.-C. Puchta, Parity patterns in one-relator groups, *J. Group Theory* **6** (2003), 245–260.

[5] T. W. Müller, J.-C. Puchta, Character theory of symmetric groups, subgroup growth of Fuchsian groups, and random walks, submitted.

[6] Y. Roichman, Upper bound on the characters of the symmetric groups, *Invent. math.* **125** (1996), 451–485.

[7] Y. Roichman, Decomposition of the conjugacy representation of the symmetric groups, *Israel J. Math.* **97** (1997), 305–316.



Key words: subgroup growth, conjugacy representation, symmetric groups, character estimates, probabilities on finite groups.

MSC-index: 20E07, 20C30, 60B15



Thomas W. Müller, School of Mathematical Sciences, Queen Mary & Westfield College, University of London, Mile End Road, E1 4NS London, UK (T.W.Muller@qmul.ac.uk)

Jan-Christoph Schlage-Puchta, Mathematisches Institut, Albert-Ludwigs-Universität, Eckerstr. 1, 79104 Freiburg, Germany (jcp@math.uni-freiburg.de)