Partitions of the unit interval generated by the Farey points

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Let $[0, 1]$ be a unit interval and $\mathcal{F}_n = \{x_{0,n}, \ldots, x_{N(n),n}\}$ be a set of $N(n) + 1$ points ordered so that $0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1$. The points $x_{0,n}, \ldots, x_{N(n),n}$ create a partition of the interval $[0, 1)$ into subintervals $[x_{i-1,n}, x_{i,n})$ of lengths $p_{i,n} = x_{i,n} - x_{i-1,n}$.

We discuss two classes of uniformity criteria of the partitions and two ways of generating the points of $\mathcal{F}_n$. The uniformity criteria are: entropy-related sums $\sigma_\beta^{(n)} = \sum_{i=1}^{N(n)} p_{i,n}^\beta$ and discrepancies

$$E_n(\alpha) = \left(\sum_{x_{i,n} \le \alpha} 1\right) - \alpha N(n), \quad D_n = \sqrt{\int_0^1 |E_n(\alpha)|^2 d\alpha}\,.$$

Here $\alpha \in (0, 1)$ and $\beta \ge 0,\ \beta \ne 1$.

We consider the following two sets of Farey points:

Farey series: $\mathcal{F}_n$ is the collection of all fractions p/q with $p \le q$, $(p, q) = 1$ and denominators $1 \le q \le n$.

Farey tree: $\mathcal{F}_n$ is the set of fractions $p/q \in [0, 1]$ such that the sum of partial quotients in their continued fraction representation is $\le n$.

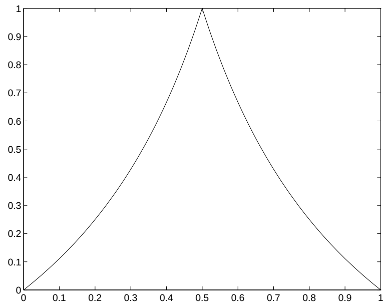
For the Farey series, it is well-known that the statements

$$E_n(\alpha) = O\left(n^{\frac{1}{2}+\varepsilon}\right),\ D_n(\alpha) = O\left(n^{\frac{1}{2}+\varepsilon}\right) \quad \forall \varepsilon > 0, \ \ (n \to \infty)$$

are equivalent to the Riemann hypothesis. I will show the graphs of $E_n(\alpha)$ and discuss some numerical results concerning finding zeroes of the Riemann zeta-function from these graphs. Computing the asymptotic behaviour of $\sigma_\beta^{(n)}$ for the Farey series is a much easier problem.

The Farey tree is closely associated with continued fractions and the Farey map $T : [0,1] \to [0,1]$ which is shown below and defined by

$$T(x) = \begin{cases} x/(1-x) & \text{if } 0 \le x < 1/2 \\ (1-x)/x & \text{if } 1/2 \le x \le 1\,. \end{cases}$$



The Farey map is almost expanding. It has the absolutely continuous invariant density $p(x) = 1/x \quad (0 < x < 1)$ and it is ergodic with respect to this density; the density $p(x)$ has infinite mass implying that the metric entropy of $T(\cdot)$ is zero. The topological pressure P_β of the Farey map is

$$P_\beta = \lim_{n\to\infty} \frac{1}{n} \log \sigma_\beta^{(n)} = \log \lambda_\beta,$$

where λ_β is the maximal eigenvalue of the transfer operator $\mathcal{L}_\beta : C[0,1] \to C[0,1]$ defined for $f \in C[0,1]$ by

$$\mathcal{L}_\beta f(x) = \sum_{y:\,T(y)=x} f(y)/|T'(y)|^\beta\,.$$

Prellberg and Slawny (*J. Statist. Phys.*, 1992, **66**, 503–514) has studied the behaviour of the pressure P_β for $\beta < 1$ and has shown that it is zero for $\beta \ge 1$. I am going to explain that for all $\beta > 1$

$$\sigma_\beta(\mathcal{F}_n) = \frac{2}{n^\beta} \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + o\left(\frac{1}{n^\beta}\right) \quad \text{as} \quad n \to \infty$$

(which complements the results of Prellberg and Slawny) and discuss the problem of computing the decay of correlations for the Farey map.