MAS115 Calculus I

Week 12

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Revision

- Improper Integrals
- Tests for Convergence/Divergence of Integrals

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Polar Coordinates

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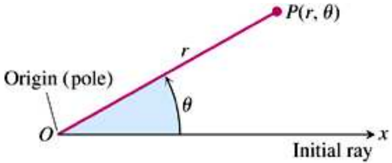

- r: the distance from the origin, O
- θ: the angle between OP and the positive x-direction

Polar Coordinates

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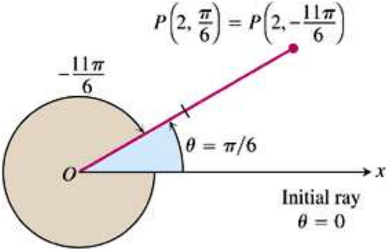

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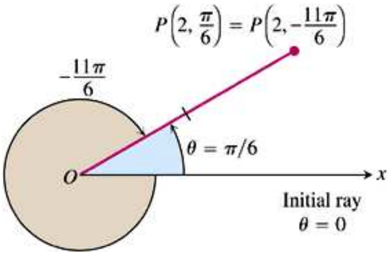

$$(r, \theta) = (r, \theta + 2\pi)$$

- if $r = 0$, the angle θ can assume any value

Polar Coordinates

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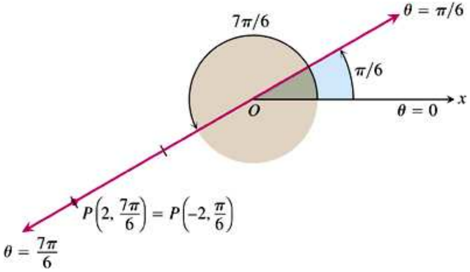
- We allow negative values for r



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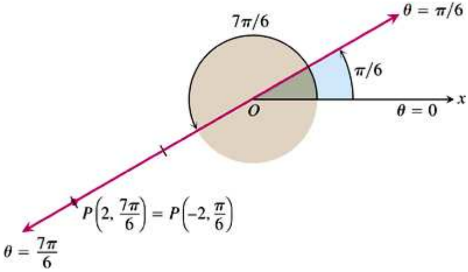

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Note: sometimes negative r is excluded (distances should not be negative), but we will find it useful for calculations.

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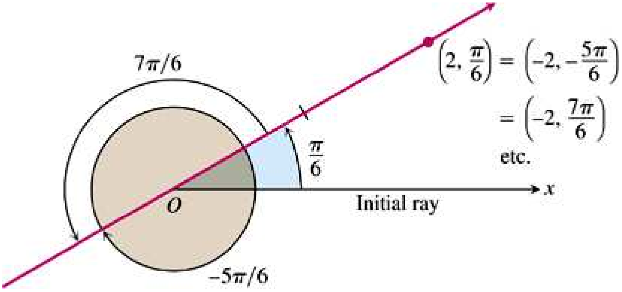

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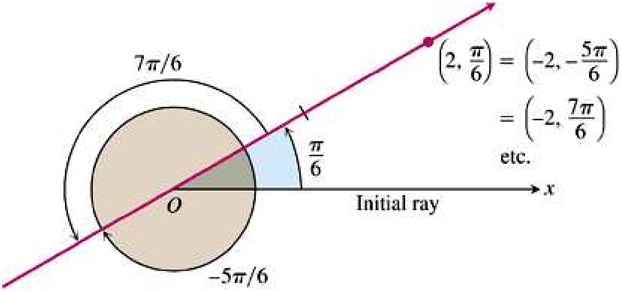

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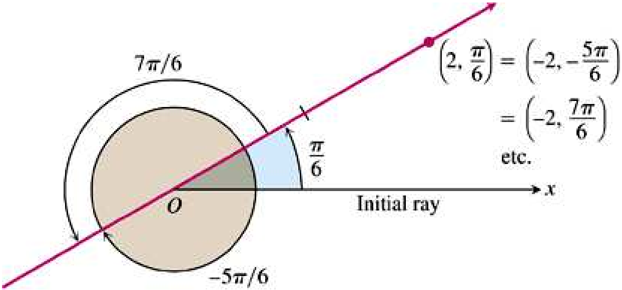

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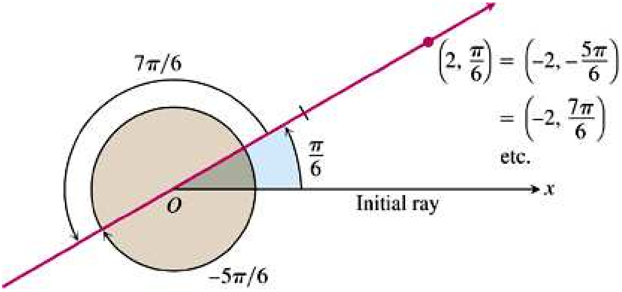

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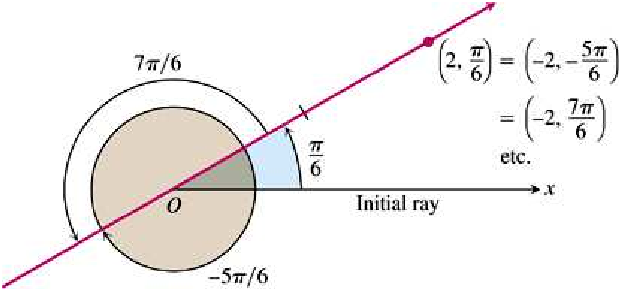

- $r = 2$: $\theta = \pi/6,\ \pi/6 \pm 2\pi,\ \pi/6 \pm 4\pi,\ \pi/6 \pm 6\pi,\ \ldots$
- $r = -2$: $\theta = 7\pi/6,\ 7\pi/6 \pm 2\pi,\ 7\pi/6 \pm 4\pi,\ 7\pi/6 \pm 6\pi,\ \ldots$

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Some graphs have simple equations in polar coordinates

- a circle about the origin:

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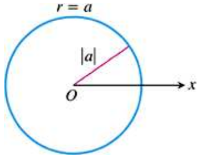

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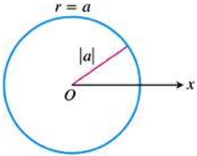

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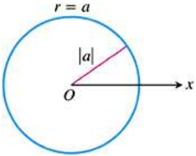

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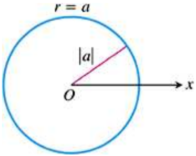

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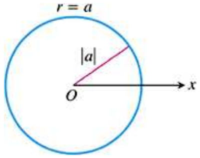

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Note: Here it becomes convenient to have allowed negative r. Otherwise the graph of $\theta = \theta_0$ would only be a ray ending at the origin.

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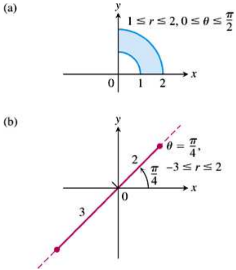
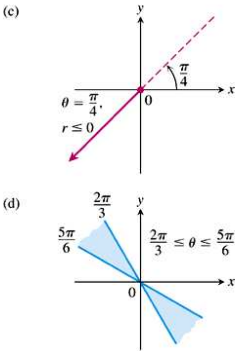
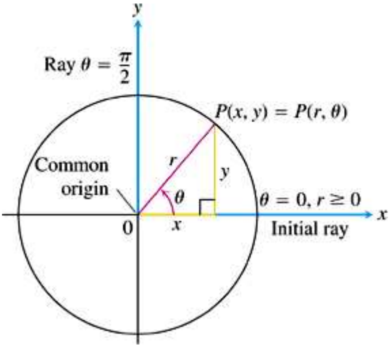




Relating Polar and Cartesian Coordinates



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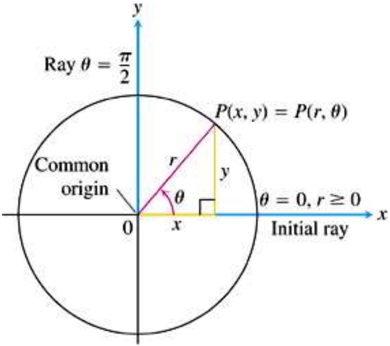

Converting polar coordinates to Cartesian coordinates:

$$x = r\cos\theta\,, \quad y = r\sin\theta$$

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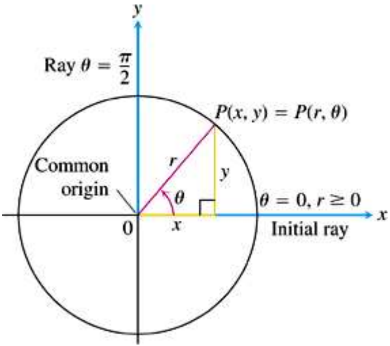

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- given (r, θ), we can uniquely compute (x, y)

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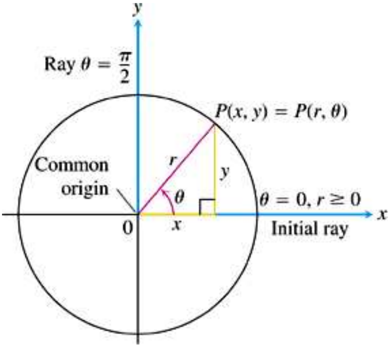

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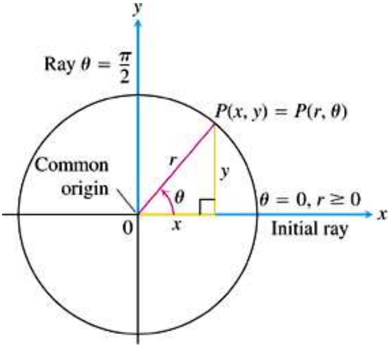

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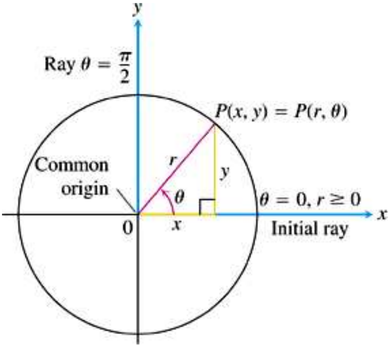

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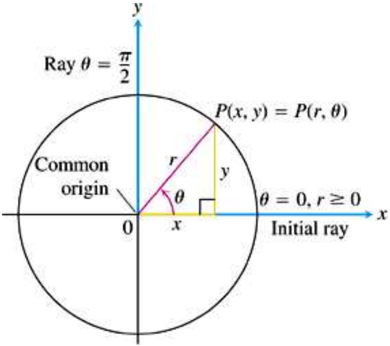

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Sometimes, polar coordinates are a lot simpler!

Converting Between Polar and Cartesian Equations

- Cartesian to polar

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- Polar to Cartesian

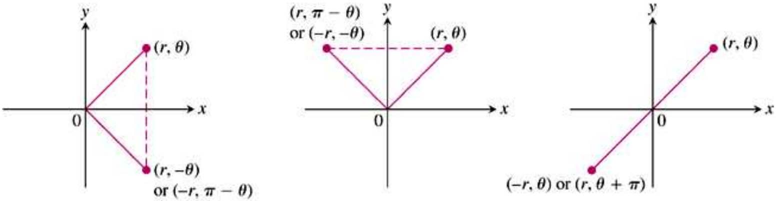
$$r = \frac{4}{2\cos\theta - \sin\theta}$$

  is equivalent to $2r\cos\theta - r\sin\theta = 4$ or $2x - y = 4$. We therefore have the equation of a line

$$y = 2x - 4$$

Symmetry in Polar Coordinates

Tests for Symmetry



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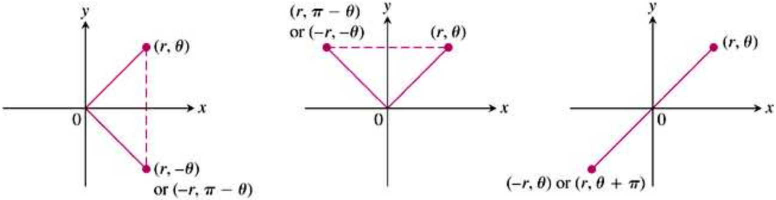

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Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph

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- Therefore

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

with

$$\begin{aligned} dx/d\theta &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ dy/d\theta &= f'(\theta)\sin\theta + f(\theta)\cos\theta \end{aligned}$$

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- A small table of values:

θ :	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r = 1 - \cos\theta$:	0	$1/2$	1	$3/2$	2

Graphing a Polar Curve

Use symmetry and monotonicity, start with table of values

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

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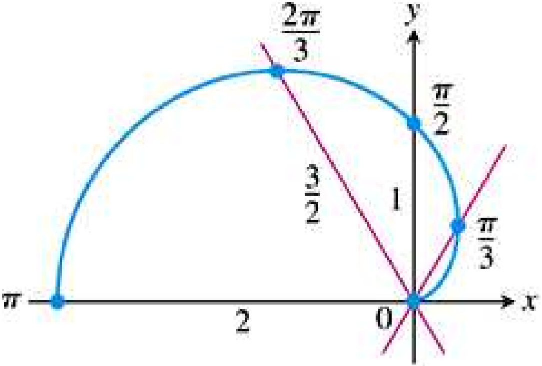

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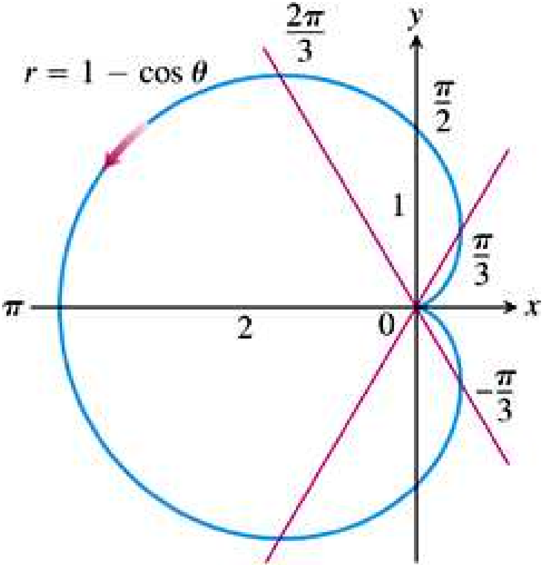

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The End