The combinatorics of the leading root of the partial theta function

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Topic Outline

Outline

Jacobi Theta Function

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$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$$

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- Quasi-periodic function satisfying

$$\vartheta(z + a + b\tau;\tau) = \exp(-\pi i b^2 \tau - 2\pi i b z)\vartheta(z;\tau)$$

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- Quasi-periodic function satisfying

$$\vartheta(z+a+b\tau;\tau) = \exp(-\pi i b^2 \tau - 2\pi i b z)\vartheta(z;\tau)$$

- Relation to modular group

$$\vartheta(z/\tau;-1/\tau) = (-i\tau)^{1/2} \exp(\pi i z^2/\tau)\vartheta(z;\tau)$$

Roots of the Jacobi Theta Function

- Jacobi Theta function

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} x^{2n}$$

where $q = \exp(\pi i \tau)$ and $x = \exp(\pi i z)$

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- Jacobi triple product

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^{2n} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1} x^2)(1 + q^{2m-1}/x^2)$$

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- Roots

$$x_k(q) = -q^{k+1/2} \qquad k \in \mathbb{Z}$$

q-Theta Function

- Combinatorialists prefer

$$\Theta(x,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}}(-x)^n = (q;q)_\infty (x;q)_\infty (q/x;q)_\infty$$

with q-product notation

$$(t;q)_n = \prod_{m=0}^{n-1}(1-tq^m)$$

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$$x_k(q)=q^k \qquad k\in\mathbb{Z}$$

Partial Theta Function

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- Roots

$$x_k(q) = ?$$

- Special case $R(x,q,0)$ of Rogers-Ramanujan Function

$$R(x,y,q) = \sum_{n=0}^{\infty} \frac{y^{\binom{n}{2}} x^n}{(q;q)_n}$$

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- Euler identities

$$R(x,1,q)=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}=\frac{1}{(x;q)_\infty}$$

and

$$R(x,q,q)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{(q;q)_n}=(-x;q)_\infty$$

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- $R(x,1,q)$ has no roots, whereas $R(x,q,q)$ has roots

$$x_k(1,q)=-q^{-k}\qquad k\in\mathbb{N}_0$$

The Leading Root of $\theta_0(x, q)$

- Consider

$$\Theta_0(x,q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n$$

and solve

$$\Theta_0(-\xi_0(q), q) = 0 \quad \xi_0(q) \in R[[q]]$$

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- The series starts

$$\xi_0(q) = 1+q+2q^2+4q^3+9q^4+21q^5+52q^6+133q^7+351q^8+948q^9+\dots$$

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- Coefficients are positive up to q^{6999}
- Similar functions seem to share such positivity properties, e.g.

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{n!}$$

Outline

Two identities for $\Theta_0(x,q)$

$\Theta_0(x,q)$ satisfies

$$\Theta_0(x,q) = (q;q)_\infty(-x;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n(-x;q)_n}$$
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- The first identity follows from Euler's identities
- The first and second identity follow from Heine's transformations for q-deformed hypergeometric functions

Proof of the first identity for $\Theta_0(x, q)$

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n = \sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n \frac{(q;q)_\infty}{(q;q)_n (q^{n+1};q)_\infty}$$

$$= (q;q)_\infty \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(q^{n+1})^m}{(q;q)_m}$$

$$= (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(xq^m)^n}{(q;q)_n}$$

$$= (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m} (-xq^m;q)_\infty$$

$$= (q;q)_\infty (-x;q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q;q)_m (-x;q)_m}$$

Two functional equations for $\xi_0(q)$

Lemma [Sokal]

$\xi_0(q)$ satisfies

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q;q)_n(\xi_0(q)q;q)_{n-1}}$$

and

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}\xi_0(q)^n}{(q;q)_n(\xi_0(q)q;q)_{n-1}}$$

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- This follows directly from the preceding identities for $\Theta_0(x, q)$

Proof of the first equation for $\xi_0(q)$

From

$$\Theta_0(x, q) = (q; q)_\infty (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (-x; q)_n}$$

it follows that

$$\Theta_0(x, q) = (q; q)_\infty (-xq; q)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (-xq; q)_{n-1}} \right]$$

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Hence $\Theta_0(-\xi_0(q), q) = 0$ implies that

$$0 = 1 - \xi_0(q) + \sum_{n=1}^{\infty} \frac{q^n}{(q;q)_n(\xi_0(q)q;q)_{n-1}}$$

A Positivity Result

Letting $\xi_0^{(0)}(q) = 1$ and iterating

$$\xi_0^{(N+1)}(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q;q)_n(\xi_0^{(N)}(q)q;q)_{n-1}}$$

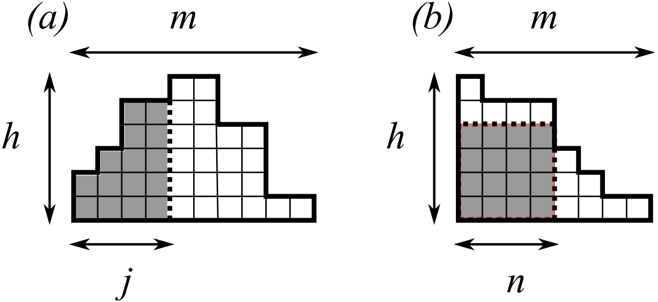
Sokal shows coefficient-wise monotonicity of $\xi_0^{(N)}(q)$, and hence positivity of $\xi_0(q)$

[A. Sokal, Adv Math, 2012]

Outline

1. Theta and Partial Theta Functions
2. Key Identities
3. Combinatorics
   - Stack Polyominoes
   - Trees Decorated with Stacks
   - Monotonicity
   - Ferrers Diagrams
   - Trees Decorated with Ferrers Diagrams
4. Many Enriched Trees
5. Outlook

Stacks and Ferrers diagrams



(a) A stack polyomino with rise j
(b) a Ferrers diagram with Durfee square of size n

Why Stack Polyominoes?

The generating function $G(x, y, a, q)$ of stack polyominoes enumerated with respect to width (x), height (y), rise (a), and total area (q), is given by

$$G(x, y, a, q) = \sum_{n=1}^{\infty} \frac{x(yq)^n}{(xq; q)_n (axq; q)_{n-1}}$$

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Compare with

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (\xi_0(q) q; q)_{n-1}}$$

to get

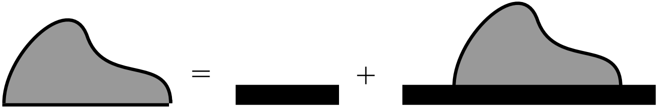
$$\xi_0(q) = 1 + G(1, 1, \xi_0(q), q)$$

Enumerating Stack Polyominoes

The result

$$G(x,y,a,q)=\sum_{n=1}^{\infty}\frac{x(yq)^n}{(xq;q)_n(axq;q)_{n-1}}$$

follows from iteration of



$$G(x) = \frac{xyq}{1-xq} + \frac{y}{(1-xq)(1-axq)}G(xq)$$

Trees decorated with Stacks

The functional equation

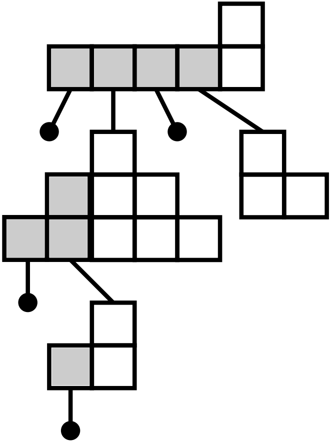
$$\xi_0(q) = 1 + G(1, 1, \xi_0(q), q)$$

admits a combinatorial interpretation using the "theory of species":

Theorem 1

Let S_q be the species of stack polyominoes augmented by the 'empty polyomino', weighted by area (q), with size given by the rise. Then $\xi_0(q)$ enumerates S_q-enriched rooted trees, weighted with respect to the total area of the stack polyominoes at the vertices of the tree.

Trees Decorated with Stacks



A Refined Enumeration

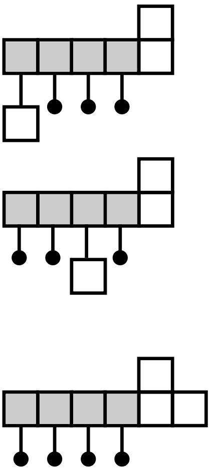
The two-variable generating function $A(t,q)$ for S_q-enriched rooted trees satisfies

$$A(t,q) = t[1 + G(1,1,A(t,q),q)]$$

where t is the generating variable for the number of vertices in the tree. One finds

$$A(t,q) = t+tq+2tq^2+(t^2+3t)q^3+(t^3+3t^2+5t)q^4+(t^4+4t^3+9t^2+7t)q^5$$
$$+(t^5+5t^4+15t^3+20t^2+11t)q^6+(t^6+6t^5+23t^4+44t^3+44t^2+15t)q^7+\dots$$

All six stack-enriched rooted trees with 5 vertices and total area 7



Monototicity of $\xi_0(q)$

The coefficients of the power series $\xi_0(q)$ are monotonically increasing.

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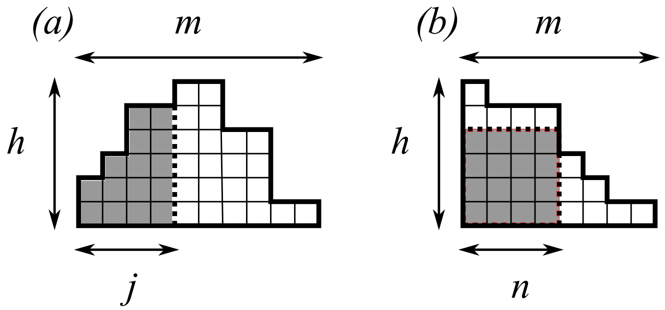
The coefficients of the power series $\xi_0(q)$ are monotonically increasing.

- Increase the area of a tree by appending a square to the bottom row of the stack at its root
- This gives an injection of trees with total area A to trees with total area $A+1$

Stacks and Ferrers diagrams



(a) A stack polyomino with rise j
(b) a Ferrers diagram with Durfee square of size n

Why Ferrers Diagrams?

The generating function $\tilde{G}(x,y,q)$ of Ferrers diagrams with n-th largest row having length n for some positive integer n, enumerated with respect to width (x), height (y), and total area (q), is given by

$$\tilde{G}(x,y,q) = \sum_{n=1}^{\infty} \frac{(xy)^n q^{n^2}}{(yq;q)_n (xq;q)_{n-1}}$$

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Compare with

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} \xi_0(q)^n}{(q;q)_n (\xi_0(q)q;q)_{n-1}}$$

to get

$$\xi_0(q) = 1 + \tilde{G}(\xi_0(q), 1, q)$$

Enumerating Ferrers Diagrams

The n-th term in the sum

$$\tilde{G}(x,y,q)=\sum_{n=1}^{\infty}\frac{(xy)^n q^{n^2}}{(yq;q)_n(xq;q)_{n-1}}$$

corresponds to a Ferrers diagram with Durfee square of size n, to which Ferrers diagrams of width $\leq n$ are appended at the top, and Ferrers diagrams of height $\leq n-1$ are appended at the right.

Trees Decorated with Ferrers Diagrams

The functional equation

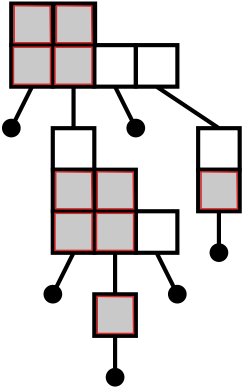
$$\xi_0(q) = 1 + \tilde{G}(\xi_0(q), 1, q)$$

admits a combinatorial interpretation using the "theory of species":

Theorem 2

Let F_q be the species of Ferrers diagrams with n-th largest row having length n for some integer n, weighted by area (q), with size given by the width of the Ferrers diagram, augmented by the 'empty polyomino'. Then $\xi_0(q)$ enumerates F_q-enriched rooted trees with respect to the total area of the Ferrers diagrams at the vertices of the tree.

Trees Decorated with Ferrers Diagrams



A Refined Enumeration

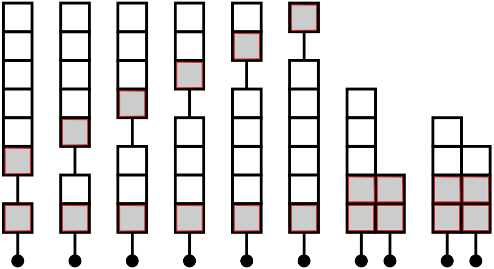
The two-variable generating function $\tilde{A}(t,q)$ for $\tilde{F}_q$-enriched rooted trees satisfies

$$\tilde{A}(t,q) = t[1 + \tilde{G}(\tilde{A}(t,q),1,q)]$$

where t is the generating variable for the number of vertices in the tree. One finds

$$\begin{aligned}\tilde{A}(t,q) &= t+t^2q+(t^3+t^2)q^2+(t^4+2t^3+t^2)q^3+(t^5+3t^4+4t^3+t^2)q^4 \\ &+ (t^6+4t^5+10t^4+5t^3+t^2)q^5+(t^7+5t^6+21t^5+17t^4+7t^3+t^2)q^6 \\ &+ (t^8+6t^7+41t^6+47t^5+29t^4+8t^3+t^2)q^7+\dots\end{aligned}$$

All eight F_q-enriched rooted trees with 3 vertices and total area 7



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Two Equinumerous Sets of Trees

S_q-enriched rooted trees with fixed total area A and F_q-enriched rooted trees with fixed total area A are equinumerous.

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- We clearly also have

$$\begin{aligned}\xi_0(q) &= F(F(\xi_0(q),q),q) = F(\tilde{F}(\xi_0(q),q),q)\\ &= \tilde{F}(F(\xi_0(q),q),q) = \tilde{F}(\tilde{F}(\xi_0(q),q),q)\end{aligned}$$

etc.
Iteration of each of these leads to a different combinatorial model

Mixed Iterations

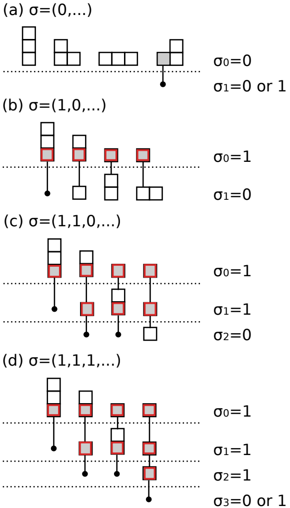
Theorem 3

Let $\sigma = \{\sigma_0, \ldots, \sigma_N\} \in \{0,1\}^{N+1}$ for $N \geq 0$. Then

$$\xi_0^\sigma(q) = F_q^{(\sigma_0)} \circ F_q^{(\sigma_1)} \circ \ldots \circ F_q^{(\sigma_N)}(0)$$

enumerates rooted trees of height at most N, enriched by S_q at level i if $\sigma_i = 0$ and enriched by F_q at level i if $\sigma_i = 1$, weighted with respect to area (level 0 is the root).
Moreover, given $\sigma \in \{0,1\}^{\mathbb{N}}$, $\xi_0(q)$ enumerates rooted trees enriched by S_q at level i if $\sigma_i = 0$ and enriched by F_q at level i if $\sigma_i = 1$, weighted with respect to area. In particular, sets of trees enriched with respect to any $\sigma \in \{0,1\}^{\mathbb{N}}$ are equinumerous for fixed total area A.

All possible trees with total area 3



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Comments and Open Problems

- Find an explicit bijection between trees decorated with stack polyominoes and Ferrers diagrams?

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- Find an explicit bijection between trees decorated with stack polyominoes and Ferrers diagrams?
- Find explicit bijections between trees generated by any two different choices of σ?
- While only monotonicity is proved rigorously, one expects a square-root singularity for $\xi_0(q)$
- Numerics indicates that the n-th coefficient of $\xi_0(q)$ grows asymptotically as

$$[q^n]\xi_0(q) \sim A\mu^n n^{-3/2} \quad \text{as } n \to \infty$$

with $\mu = 3.2336366652450763163646925293 87\ldots$