Combinatorial Enumeration with the Kernel Method

1. Background (personal interest, no Kernel Method)

Want to understand classes of (self-avoiding) walks
- Mathematics: counting formulas, generating functions, ...
- Physics: lattice models of polymers, add weights

to study collapse, adsorption, ...

Questions: thermodynamic limit, phase transitions, critical exponents, ...

Example 1 Self-avoiding walks on $\mathbb{Z}^2$ (SAW)

C_N = # of N-step walks starting at O

hard combinatorial question

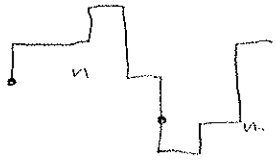
Proposition $\lim_{N\to\infty} C_N^{1/N} = \mu_{SAW}$ exists

Lemma (Subadditivity) If $a_{n+m} \leq a_n + a_m$

then $\lim_{n\to\infty} \frac{1}{n} a_n = \inf_{n\to\infty} \frac{1}{n} a_n$.

Remark: may be $-\infty$, need lower bound to prove finite limit

Proof (of proposition)



Concatenation: $c_{n+m} \leq c_n \cdot c_m$

take $a_n = \log c_n \rightsquigarrow a_{n+m} \leq a_n + a_m \rightsquigarrow \lim_{n\to\infty} \frac{1}{n} a_n = \inf_{n\to\infty} \frac{1}{n} a_n$

$= \log \mu$

$c_n \geq 2^n$ (directed walks $\uparrow\!\rightarrow$)

$\rightsquigarrow \quad 2 \leq \mu_{SAW}$

$c_n \leq 3^n$ (random walks w/o reversal)

$\rightsquigarrow \quad \mu_{SAW} \leq 3$

Remarks:
- refinement gives $c_N = \mu_{SAW}^{N + O(N^{1/2})}$
- known that $\frac{c_{N+2}}{c_N} \to \mu^2$, but $\frac{c_{N+1}}{c_N} \to \mu$ is open problem
- Physicists "know" that $c_N = \mu_{SAW}^N N^{\gamma-1} A \, (1+o(1))$

  with $\gamma = 43/32$ (exact, non-rigorous)

  $\mu_{SAW} = 2.68\ldots$ estimated up to 16 decimal places

- important connection to stochastic Loewner equation $SLE_{8/3}$ would lead to rigorous results if the scaling limit of SAW could be shown to be "conformally invariant".

<u>Example 1a</u> self-avoiding polygons on $\mathbb{Z}^2$ (SAP)

(counting translation-invariant equivalence classes)

$$p_N = \mu_{SAP}^N N^{\alpha-2} B (1+o(1))$$

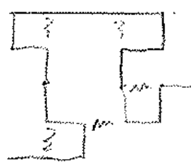
- $\mu_{SAP} = \mu_{SAW}$ (rigorous)
- $\alpha = \frac{1}{2}$ (non-rigorous)

<u>Exercise</u> prove existence of μ_{SAP}, give bounds

<u>Example 2</u> interacting self-avoiding walks on $\mathbb{Z}^2$ (ISAW)

$c_{N,M}$ = # of N-step SAW with M interactions

(non-consecutive nearest-neighbours)



construct generating function (partition function)

$$Z_N(\omega) = \sum_m c_{N,M}\, \omega^M$$

$$\lim_{N\to\infty} \frac{1}{N} Z_N(\omega) = K(\omega)$$

<u>translation to physics:</u> $\omega = e^{-\beta J}$ Boltzmann weight
where $\beta = \frac{1}{k_B T}$ with T temperature, k_B Boltzmann constant.

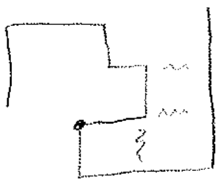
weight of a walk with m interactions has energy $E = mJ$:

$$\omega^m = e^{-\beta m J} = e^{-\beta E}$$

attractive interactions: $J<0$, $E<0$, $\omega>1$.

free energy $f(T)$: $K(\omega) = -\frac{1}{k_B T} f(T)$, $\log \omega = -\frac{1}{k_B T} J$

existence of $K(\omega)$ is proved for $\omega \leq 1$:



$$Z_{N+M}(\omega) \leq Z_N(\omega) Z_M(\omega)$$

proof breaks down for $\omega > 1$ (self-avoidance & interactions have opposite effect)

$\rightsquigarrow$ $K(\omega)$ not rigorously known to exist.

<u>Exercise:</u> prove existence of $K(\omega)$ for interacting SAP for $\omega > 0$

Physicists "know" much more: $\exists\, \omega_c$ such that

$$Z_N(\omega) \sim A(\omega)\mu(\omega)^N N^{\gamma-1} \quad \text{for } \omega < \omega_c$$

$$Z_N(\omega_c) \sim A(\omega_c)\mu(\omega_c)^N N^{\gamma_t-1} \quad \text{for } \omega = \omega_c$$

$$Z_N(\omega) \sim A(\omega)\mu(\omega)^N \mu_s(\omega)^{N^{1/2}} N^{\gamma_c-1} \quad \text{for } \omega > \omega_c$$

$\mu(\omega)$ is continuous for $\omega \geq 0$

$\mu(\omega)$ is real-analytic for $\omega < \omega_c$ and $\omega > \omega_c$

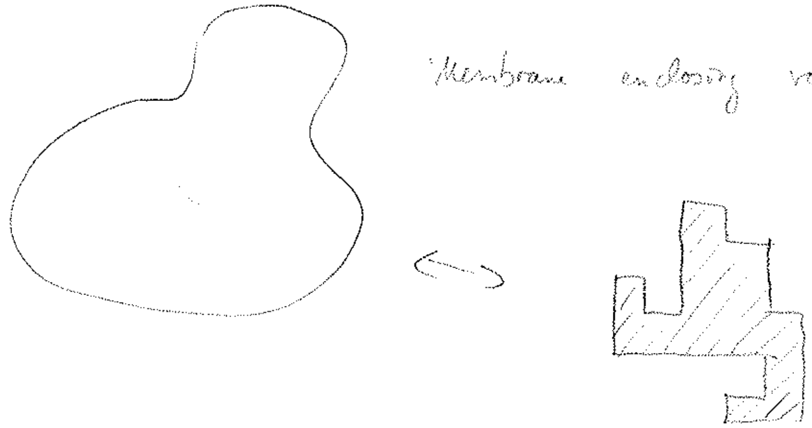
"<u>PHASE TRANSITION</u>" at $\omega = \omega_c$ (in literature: θ-point)

moreover: intricate cross-over behaviour for ω near ω_c.

Huge gap between physicists' knowledge and mathematicians' rigour.

$\rightsquigarrow$ Need for alternative walk models that can be analyzed rigorously.

<u>Example 3</u> Lattice models of vesicles (vesiculum = bubble)



$c_{N,M}$ # of SAPs with perimeter N enclosing area M

$$Z_N(q) = \sum_M c_{N,M}\, q^M \qquad \text{finite-perimeter part. fction}$$

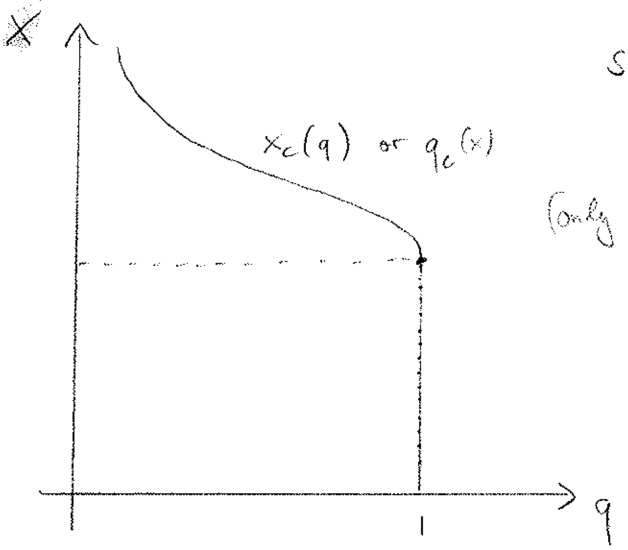
$$Q_M(x) = \sum_N c_{N,M}\, x^N \qquad \text{finite-area part. function}$$

or

$$G(x,q) = \sum_{N,M} c_{N,M}\, x^N q^M \qquad \text{"grand-canonical" partition function}$$

$$= \sum_N Z_N(q)\, x^N = \sum_M Q_M(x)\, q^M$$

Singularity diagram



(only closest singularity to the origin)

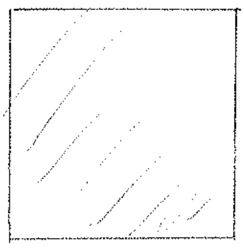
$$\lim_{N\to\infty} Z_N^{1/N}(q) = \frac{1}{x_c(q)} \qquad \lim_{m\to\infty} Q_m^{1/m}(x) = \frac{1}{q_c(x)}$$

(N even)

Exercise: prove existence of $x_c(q)$ for SAP with $q>0$

$$x_c(1) = \frac{1}{\mu_{SAW}}$$

$x_c(q) = 0$ for $q > 1$:



n

$$Z_{4n}(q) \geq q^{n^2}$$

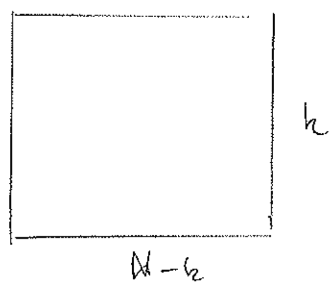
jump of $x_c(q)$ at $q=1$: phase transition

<u>Theorem</u>

$$Z_{2N}(q) = \frac{1}{\prod_{k=1}^{\infty}(1-q^{-k})^4} \sum_{k=-\infty}^{\infty}{}' q^{k(N-k)} \left(1+O(\rho^{+N})\right)$$

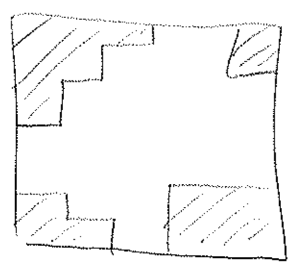
for some $0<\rho<1$

<u>Idea of proof</u>: for $q>1$, dominating polygons are close to rectangles:



$$R_N(q) = \sum_{k=1}^{N-1}{}' q^{k(N-k)}$$

corrections to $R_N(q)$ come from "missing corners".



GF for corner is area-GF for Ferrer's diagram $\frac{1}{\prod_{k=1}^{\infty}(1-q^{k})} = \frac{1}{(q;q)_\infty}$

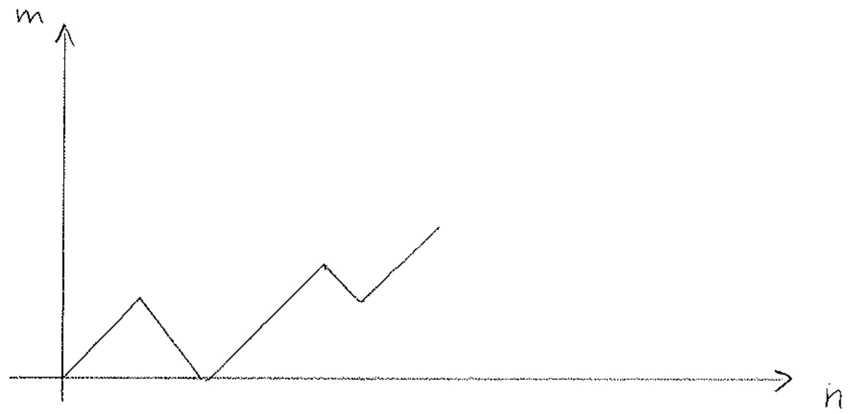
removing four corners (ignoring overlaps) does not change the perimeter: multiply by $\frac{1}{(q^{-1};q^{-1})_\infty^4}$

$$Z_{2N}(q) \approx \frac{1}{(q^{-1};q^{-1})_\infty^4} \sum_{k=\cancel{1}\,-\infty}^{\cancel{N-1}\,\infty}{}' q^{k(N-k)} .$$

← exp. small error

The rest is hard estimates.

2. Yet another enumeration of Dyck paths (walks on $\mathbb{N}_0$)



$C_{N,M}$ # of N-step walks ending at position M

- recurrence (semi-infinite transfer matrix) $\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$

  + related tricks

- reflection principle

- here: functional equation

$$G(x,t) = \sum_{N,M} C_{N,M}\, t^N x^M$$

$$C_{0,M} = \delta_{M,0} \qquad C_{N+1,M} = \begin{cases} C_{N,M-1} + C_{N,M+1} & M > 0 \\ C_{N,M+1} & M = 0 \end{cases}$$

leads to

$$G(x,t) = 1 + t\left(x\, G(x,t) + \frac{1}{x}\, G(x,t)\right)$$

$$- t\, \frac{1}{x}\, G(0,t)$$

better: correction of overcounting

no boundary:

$$G_F(x,t) = 1 + t\left(x + \frac{1}{x}\right) G_F(x,t) \quad \rightsquigarrow \quad G_F(x,t) = \frac{1}{1 - t(x+\frac{1}{x})}$$

correct for steps crossing the boundary: $\quad - t \frac{1}{x} G(0,t)$

$$\left[1 - t\left(x + \frac{1}{x}\right)\right] G(x,t) = 1 - \frac{t}{x} G(0,t)$$

$$G(x,t) = \frac{1 - \frac{t}{x} G(0,t)}{1 - t(x + \frac{1}{x})}$$ so $G(0,t)$ determines $G(x,t)$

- cannot simply plug in $x = 0$
- factor $1 - t(x + \frac{1}{x}) = -\frac{t}{x}\left(x - x_0(t)\right)\left(x - x_1(t)\right)$
- $x^2 - \frac{1}{t}x + 1 = 0 \quad \rightsquigarrow \quad x_{0,1} = \frac{1}{2t}\left(1 \pm \sqrt{1 - 4t^2}\right)$ $\qquad x_0 x_1 = 1$, $\quad x_0 + x_1 = \frac{1}{t}$

so if $x \to x_0$ or $x \to x_1$ then (hopefully)

$$\lim_{x \to x_i} \left[1 - t\left(x + \frac{1}{x}\right)\right] G(x,t) = 0 \quad \text{and} \quad 0 = 1 - \frac{t}{x_i} G(0,t)$$

so $G(0,t) = \frac{x_i}{t}$ and $x_i = x_1$: $\frac{x_1}{t} = \frac{1}{2t^2}\left(1 - \sqrt{1 - 4t^2}\right) = C(t^2)$

whereas $x_i = x_0$: $\frac{x_0}{t} = \frac{1}{2t^2}\left(1 + \sqrt{1 - 4t^2}\right) = O\left(\frac{1}{t^2}\right)$

$\left[\ C(t) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{t^n}{n+1}\right.$ Catalan GF solving $\left. C(t) = 1 + t\,C(t)^2\ \right]$

$$G(0,t) = \frac{x_1}{t} \quad \text{and therefore}$$

$$G(x,t) = \frac{1 - \frac{t}{x} G(0,t)}{1 - t(x + \frac{1}{x})} = \frac{1 - \frac{x_1}{x}}{-\frac{t}{x}(x-x_0)(x-x_1)} = \frac{1}{t(x_0 - x)}$$

GF for walks w/o boundary

$$G_F(1,t) = \frac{1}{1-2t} \quad \rightsquigarrow \quad c_N = 2^N$$

GF for walks with boundary

$$G(0,t) = C(t^2) \quad \rightsquigarrow \quad C_{2N} \sim C_1 \frac{4^N}{N^{3/2}}$$

$$G(1,t) = \frac{1 - t\,C(t^2)}{1-2t} \quad \rightsquigarrow \quad C_N \sim C_1 \frac{2^N}{N^{1/2}}$$

(C_1 "generic")

Note: Walks w/o boundary ending at 0 cannot be computed by substituting $x=0$. We need

<u>Exercise:</u>

$$[x^0] \frac{1}{1-t(x+\frac{1}{x})} = CT_x \frac{1}{1-t(x+\frac{1}{x})} = \frac{1}{2\pi i} \oint \frac{1}{-\frac{t}{x}(x-x_0)(x-x_1)} \frac{dx}{x}$$

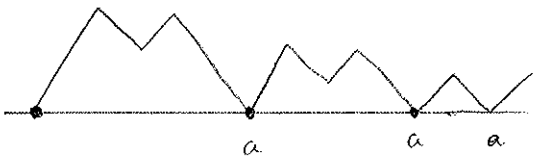
$$= \frac{1}{2\pi i} \oint \frac{1}{\sqrt{1-4t^2}} \left(\frac{1}{x-x_1} - \frac{1}{x-x_0} \right) dx = \frac{1}{\sqrt{1-4t^2}}, \quad C_{2N} \sim C_1 \frac{4^N}{N^{1/2}}$$

An application: adsorption of directed polymers

weigh contacts with boundary with weight a

$$G(x,a,t) = 1 + t\left(x+\frac{1}{x}\right) G(x,a,t) - \frac{t}{x} G(0,a,t)$$

$$\underline{\underline{+\,tx(a-1)\;G(0,a,t)}}$$

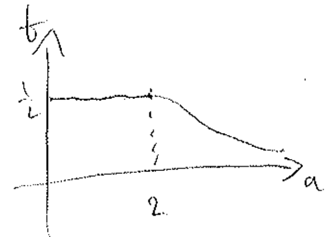


$$\underbrace{\left[1-t\left(x+\frac{1}{x}\right)\right]}_{K(x_1,t)=0} G(x,a,t) = 1 - t\left(\frac{1}{x} + x(1-a)\right) G(0,a,t)$$

$$\rightsquigarrow\; G(0,a,t) = \frac{x_1}{t\left(1+x_1^2(1-a)\right)} = \frac{C(t^2)}{t\left(1-a\right)\left(C(t^2)-1\right)}$$

- square-root singularity at $t = \frac{1}{2}$ for $a<2$ $\qquad Z_N \sim 2^N N^{-3/2}$
- pole at $C(t^2) = \frac{a}{a-1}$ for $a>2$ $\qquad Z_N \sim \mu^N$
- $\frac{1}{\text{square-root}}$ singularity for $a=2$ $\qquad Z_N \sim 2^N N^{-1/2}$

$\rightsquigarrow$ polymer adsorbs at $a_c = 2$, phase transition



3. A trick used twice becomes a method:

We've just encountered the Kernel Method for

a functional equation of the form (dropping t)

$$K(x)\, G(x) = F(x, G(0))$$

Often, one isn't interested in the variable x, but only in special values ($x=0$, or $x=1$, say), but varying x is essential for solving the eqn. The variable x is called catalytic. $K(x)$ is called the Kernel.

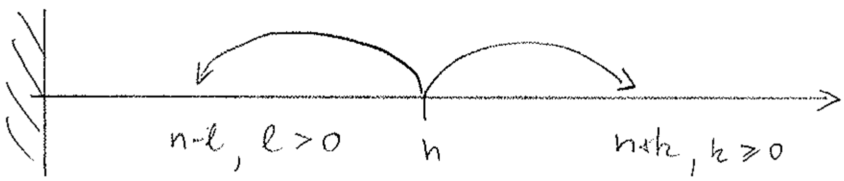
Method: (i) solve $K(x)=0 \rightsquigarrow x = X$

(ii) solving $F(X, G(0))$ determines $G(0)$

Origin: Knuth TACP I 1968 (exercise)

Prodinger '99: "The French have a new toy. They call it the Kernel Method"

<u>Kernel method for a larger class of walks</u>



$\ell \in A$ $\qquad\qquad$ $k \in B$

$A(x) = \sum_{\ell \in A}' x^\ell$

$B(x) = \sum_{k \in B}' x^k$

allow finitely many forward and backward jumps

$\deg A = a$

$\deg B = b$

$$G(x,t) = \underset{\substack{\uparrow \\ \text{no walk}}}{1} + t\left(A(x) + B\left(\tfrac{1}{x}\right)\right) G(x,t)$$

$$- t\left[B\left(\tfrac{1}{x}\right) G(x,t)\right]_{<0}$$

$$= 1 + t\left(A(x) + B\left(\tfrac{1}{x}\right)\right) G(x,t)$$

$$- t \sum_{k=0}^{b-1}{}' b_k\left(\tfrac{1}{x}\right) G_k(t)$$

rewrite

$$\underbrace{\left[1 - t\left(A(x) + B\left(\tfrac{1}{x}\right)\right)\right] x^b}_{K(x,t)} G(x,t) = \underbrace{x^b\left(1 - t\sum_{k=0}^{b-1}{}' b_k\left(\tfrac{1}{x}\right) G_k(t)\right)}_{R(x,t)}$$

<u>exercise</u>: write functional equation for $A = \{0,1\}$ and $B = \{1,2,3\}$

↭

$$K(x,t) = x^b\left(1 - t\left[A(x) + B\left(\tfrac{1}{x}\right)\right]\right)$$

has degree $a+b$ in x and admits $a+b$ solutions as algebraic functions of t.

- b "small" branches $x = u_i \sim \omega t^{1/b}$ $\qquad \omega^b = 1$
- a "large" branches $x = v_i \sim \omega t^{-1/a}$ $\qquad \omega^a = 1$

[Puiseaux expansion]

$$K(x,t) = -t \prod_{i=1}^{b} (x-u_i) \prod_{i=1}^{a} (x-v_i)$$

$R(x,t)$ is a polynomial of degree b in x, therefore necessarily

$$R(x,t) = \prod_{i=1}^{b} (x-u_i) \qquad \left[\text{ no need to work with } G_k(t)! \right]$$

Therefore we find

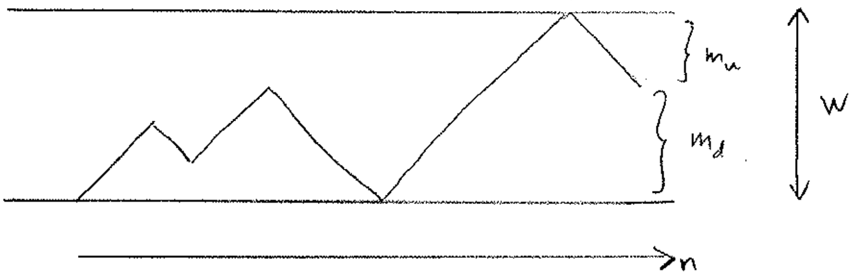
$$G(x,t) = -\frac{1}{t \prod_{i=1}^{a} (x-v_i)}$$

and

$$G(0,t) = \frac{(-1)^{a-1}}{t \prod_{i=1}^{a} v_i} \quad , \quad G(1,t) = -\frac{1}{t \prod_{i=1}^{a} (1-v_i)}$$

$$\left[\text{compare Dyck: } \quad G(0,t) = \frac{1}{t x_0} = \frac{x_1}{t} \quad , \quad G(1,t) = -\frac{1}{t(1-x_0)} \right]$$

4. Directed walks in a strip



$$G(\underset{\substack{\uparrow\\ m_d}}{x},\underset{\substack{\uparrow\\ m_u}}{y},\underset{\substack{\uparrow\\ n}}{t}) = y^W + t\left(\frac{x}{y}+\frac{y}{x}\right) G(x,y,t)$$

$$- t\,\frac{x}{y}\,G(x,0,t) - t\,\frac{y}{x}\,G(0,y,t)$$

rewrite: $$xy\left(1-t\left(\frac{x}{y}+\frac{y}{x}\right)\right)G(x,y,t) = xy^{W+1} - tx^2 G(x,0,t)$$

$$-ty^2 G(0,y,t)$$

or $$K(x,y,t)\,xy\,G(x,y,t) = xy^{W+1} - R(x,t) - S(y,t)$$

$K(x,y,t)=0 \;\rightsquigarrow\; y=y(x,t)$ relates y and x

think of <u>pairs</u> (x,y) killing the kernel. Here, $y=qx$

where $q+\frac{1}{q}=\frac{1}{t}$, so if $K(x,qx,t)=0$ then

K vanishes also for $(x,qx),\ (q^2x,qx),\ (q^2x,q^3x),\ \ldots$

So iterate: $$R(x,t) = x(qx)^{W+1} - S(qx,t)$$

$$S(qx,t) = q^2x(qx)^{W+1} - R(q^2x,t) \quad \text{etc}$$

$$R(x,t) = x^{w+2} q^{w+1} - x^{w+2} q^{w+3} + R(q^2 x, t)$$

$$= x^{w+2} q^{w+1}(1-q^2) + R(q^2 x, t)$$

$$= \ldots = \frac{x^{w+2} q^{w+1} (1-q^2)}{1-q^{2(w+2)}} \qquad \text{(geom series)}$$

Therefore $$G(x,0,t) = x^w \frac{q^{w+1}(1-q^2)}{t(1-q^{2(w+2)})} = x^w \frac{q^w(1-q^4)}{1-q^{2(w+2)}}$$

<u>exercise:</u> compute $$G(0,y,t) \left[= y^w \frac{(1+q^2)(1-q^{2(w+1)})}{1-q^{2(w+2)}} \right]$$

and $$G(1,1,t) \left[= \frac{(1+q^2)(1-q^{w+1})(1-q^{w+2})}{(1-q)(1-q^{2(w+2)})} \right]$$

Generalize to contact weights a/b at bottom / top:

e.g. $$G_{ab}(x,0,t) = x^w \frac{b\, q^w (1-q^4)}{(1+(a-1)q^2)(1-(b-1)q^2) - ((1-a)+q^2)((1-b)+q^2)q^{2w}}$$

5. 2-dimensional lattice walks I: walks on the slit plane

C_{N,M_x,M_y}, # of N-step walks from 0 to (M_x, M_y)

$$G(x,y,t) = \sum_{N,M_x,M_y} c_{N,M_x,M_y}\, t^N x^{M_x} y^{M_y}$$, step set Ω:

- no boundaries: $$G_F(x,y,t) = 1 + t \sum_{\sigma\in\Omega} x^{\sigma_x} y^{\sigma_y}\, G_F(x,y,t)$$
  $$\rightsquigarrow G_F(x,y,t) = \left[K(x,y,t)\right]^{-1}$$
- walks in the slit plane (starting at 0 but must not return to $(-n,0)$, $n \in \mathbb{N}_0$)
  $\Omega = \{\uparrow, \leftarrow, \downarrow, \rightarrow\}$

$$G(x,y,t) = 1 + t\left(x + \frac{1}{x} + y + \frac{1}{y}\right) G(x,y,t) - B\left(\frac{1}{x}, t\right)$$

walks starting at 0, avoiding slit but ending on it.

$$\left[1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right] G(x,y,t) = 1 - B\left(\frac{1}{x}, t\right)$$

careful: mindless application of the method gives nonsense:

$LHS = 0 \rightsquigarrow RHS = 0 \rightsquigarrow B\left(\frac{1}{x}\right) = 1$ does not make sense!

What's wrong? With the power of hindsight,

$c_N \sim 4^N N^{-1/4}$ so that $G(x,y,t)$ diverges so fast that

$$\lim_{y\to y(x,t)} \left[1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right] G(x,y,t) \neq 0$$

Repair this by considering instead of $G(x,y,t) = \sum_M y^M G_M(x,t)$

$$H(x,y,t) = \sum c_{N,M_x,M_y} t^N x^{M_x} y^{|M_y|} = \sum_M y^{|M|} G_M(x,t)$$

$$[G_M = G_{-M}]$$

$$H(x,y,t) = 1 + t\left(x+\frac{1}{x}+y+\frac{1}{y}\right) H(x,y,t) + t\left(y-\frac{1}{y}\right) G_0(x,t) - B\left(\frac{1}{x},t\right)$$

walks ending on x-axis

so that

$$(*) \quad \underbrace{\left[1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right]}_{K(x,y,t)} H(x,y,t) = 1 + t\left(y-\frac{1}{y}\right) G_0(x,t) - B\left(\frac{1}{x},t\right)$$

$$K(x,y,t) = 0 \quad \rightsquigarrow \quad y^2 - \left(\frac{1}{t} - \left(x+\frac{1}{x}\right)\right) y + 1$$

two roots $Y_0 Y_1 = 1$ and $Y_1 = Y_1(x,t) = \frac{1}{2}\left(\frac{1}{t} - \left(x+\frac{1}{x}\right)\right) - \sqrt{\frac{1}{4}\left(\frac{1}{t}-\left(x+\frac{1}{x}\right)\right)^2 - 1}$

$Y_0 \sim \frac{1}{t}$ $\quad Y_1 \sim t$ with coefficients Laurent pols in x.

Substitute $y = Y_1$ into $(*)$ to get

$$1 - B\left(\tfrac{1}{x}, t\right) = t\left(\frac{1}{Y_1} - Y_1\right) G_0(x,t) \qquad \left[\text{compare } 1 - B\left(\tfrac{1}{x}, t\right) = 0\right]$$

$$= 2t\sqrt{\frac{1}{4}\left(\frac{1}{t} - x - \frac{1}{x}\right)^2 - 1}\; G_0(x,t)$$

$$\underbrace{1 - B\left(\tfrac{1}{x}, t\right)}_{\text{non-pos. powers in } x} = \sqrt{\left(1 - t\left(x + \tfrac{1}{x}\right)\right)^2 - 4t^2}\; \underbrace{G_0(x,t)}_{\text{non-neg powers in } x}$$

__Trick:__ Factorize $\left(1 - t\left(x + \frac{1}{x}\right)\right)^2 - 4t^2 = \left(1 - t\left(x + \frac{1}{x} + 2\right)\right)\left(1 - t\left(x + \frac{1}{x} - 2\right)\right)$

$$= D(t)\,\Delta(x,t)\,\Delta\left(\tfrac{1}{x}, t\right)$$

where $\Delta(x,t) = \left(1 - x\left(C(t) - 1\right)\right)\left(1 - x\left(1 - C(-t)\right)\right)$

and $D(t) = \left[C(t)\,C(-t)\right]^{-2}$

__exercise__: confirm factorization

now
$$\frac{1 - B\left(\frac{1}{x}, t\right)}{\sqrt{\Delta\left(\frac{1}{x}, t\right)}} = \sqrt{D(t)}\;\sqrt{\Delta(x,t)}\;G_0(x,t)$$

LHS non-pos powers in x

RHS non-neg powers in x

LHS = RHS must be constant independent of x

$\Delta(0,t) = 1$, $G_0(0,t) = 1$ [walks cannot return to 0]

so that
$$\frac{1 - B(\frac{1}{x}, t)}{\Delta(\frac{1}{x}, t)} = \sqrt{D(t)}$$
, or

$$1 - B(\tfrac{1}{x}, t) = \sqrt{D(t)\, \Delta(\tfrac{1}{x}, t)}$$

and thus

$$G(x,y,t) = \frac{\sqrt{D(t)\, \Delta(\frac{1}{x}, t)}}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}$$

$$G(1,1,t) = \frac{\left(1 + \sqrt{1+4t}\right)^{1/2} \left(1 + \sqrt{1-4t}\right)^{1/2}}{2(1-4t)^{3/4}}$$

$$c_n \sim \frac{\sqrt{1+\sqrt{2}}}{2\,\Gamma(3/4)}\, 4^n\, n^{-1/4}$$

trick to method: <u>The factorisation lemma</u>

Let $\mathcal{J}(x,t)$ be a polynomial in t with coefficients in $\mathbb{R}[x,\bar{x}]$ and assume $\mathcal{J}(x,0)=1$. $\quad (\bar{x}=\frac{1}{x})$

There exist a unique triple $(\mathcal{D}(t), \Delta(x,t), \bar{\Delta}(\bar{x},t))$ of FPS in t satisfying

- $\mathcal{J}(x,t) = \mathcal{D}(t)\cdot\Delta(x,t)\cdot\bar{\Delta}(\bar{x},t)$
- coeffs of $\mathcal{D}(t)$ belong to $\mathbb{R}$
- " $\Delta(x)$ " $\mathbb{R}[x]$
- " $\bar{\Delta}(\bar{x})$ " $\mathbb{R}[\bar{x}]$
- $\mathcal{D}(0) = \Delta(0,t) = \bar{\Delta}(0,t) = \Delta(x,0) = \bar{\Delta}(\bar{x},0) = 1$

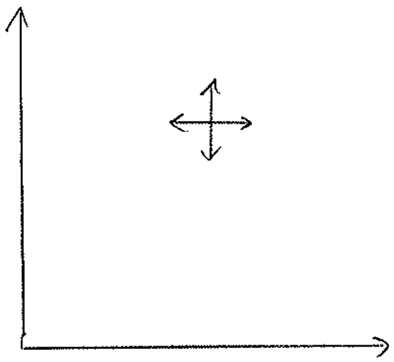
Moreover, these three series are algebraic, and $\Delta(x)$ is a pol in x
$(\bar{\Delta}(\bar{x}))$ $\qquad (\bar{x})$

Change steps: $\longrightarrow$ $\nwarrow\nearrow\swarrow\searrow$

$$\left[1-t\left(\frac{x}{y}+\frac{y}{x}+xy+\frac{1}{xy}\right)\right]H(x,y,t) = 1+t\left(xy+\frac{y}{x}-\frac{x}{y}-\frac{1}{xy}\right)G_0(x,t)$$
$$-B\left(\frac{1}{x},t\right)$$

<u>Exercise</u>: find $\mathcal{J}(x,t)$, compute factorisation

6. 2-dimensional lattice walks II: walks on the quarter plane



$$G(x,y,t) = 1 + t\left(x+y+\frac{1}{x}+\frac{1}{y}\right) G(x,y,t) - t\frac{1}{x}\, G(0,y,t) - t\frac{1}{y}\, G(x,0,t)$$

$$\rightsquigarrow K(x,y,t)\,xy\,G(x,y,t) = xy - t\,x\,G(x,0,t) - t\,y\,G(0,y,t)$$

Note that $K(x,y,t) = 1 - t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)$ has symmetries $x \leftrightarrow \frac{1}{x}$, $y \leftrightarrow \frac{1}{y}$, $x \leftrightarrow y$

(1) $K(x,y,t) = 0 \quad \rightsquigarrow \quad Y_1 = f_t\left(x+\frac{1}{x}\right) \sim t \qquad Y_0 = \frac{1}{Y_1} \sim \frac{1}{t}$

iterate: $(x, Y_1) \to \left(\frac{1}{x}, Y_1\right) \to \left(\frac{1}{x}, \frac{1}{Y_1}\right) \to \left(x, \frac{1}{Y_1}\right) \to (x, Y_1)$

✓ ✓ ↑ ↑

no powerseries int.: not admissible

$K(x,Y_1) = 0 \quad \rightsquigarrow \quad xY_1 = tx\,G(x,0,t) - tY_1\,G(0,Y_1,t)$

$K\left(\frac{1}{x},Y_1\right) = 0 \quad \rightsquigarrow \quad \frac{1}{x}Y_1 = t\frac{1}{x}\,G\left(\frac{1}{x},0,t\right) - tY_1\,G(0,Y_1,t)$

so that $t\left[x\,G(x,0,t) - \frac{1}{x}\,G\left(\frac{1}{x},0,t\right)\right] = \left(x-\frac{1}{x}\right)Y_1$

and therefore

$$G(x,0,t) = \frac{1}{tx}\left[\left(x-\frac{1}{x}\right)Y_1\right]_{\geq x} \qquad \text{positive part (in } x\text{)}$$

$$= \frac{1}{t}\,\frac{1}{2\pi i}\oint_{|z|=1}\left(z-\frac{1}{z}\right) f_t\left(z+\frac{1}{z}\right)\frac{dz}{z(z-x)}$$

(2) "algebraic" Kernel method : <u>only</u> use Kernel symmetries

(don't set $K(x,y,t)=0$, just eliminate bdry terms)

<u>exercise?</u>

$$K(x,y,t)\, xy\, G(x,y,t) = xy - tx\, G(x,0,t) - ty\, G(0,y,t) \qquad +$$

$$x\to \tfrac{1}{x}: \quad K(x,y,t)\, \tfrac{1}{x}y\, G(\tfrac{1}{x},y,t) = \tfrac{1}{x}y - t\tfrac{1}{x}\, G(\tfrac{1}{x},0,t) - ty\, G(0,y,t) \qquad -$$

$$y\to \tfrac{1}{y}: \quad K(x,y,t)\, x\tfrac{1}{y}\, G(x,\tfrac{1}{y},t) = x\tfrac{1}{y} - tx\, G(x,0,t) - t\tfrac{1}{y}\, G(0,\tfrac{1}{y},t) \qquad -$$

$$K(x,y,t)\, \tfrac{1}{xy}\, G(\tfrac{1}{x},\tfrac{1}{y},t) = \tfrac{1}{xy} - \tfrac{t}{x}\, G(\tfrac{1}{x},0,t) - t\tfrac{1}{y}\, G(0,\tfrac{1}{y},t) \qquad +$$

$$K(x,y,t)\left[xy\, G(x,y,t) - \tfrac{1}{x}y\, G(\tfrac{1}{x},y,t) - x\tfrac{1}{y}\, G(x,\tfrac{1}{y},t) + \tfrac{1}{xy}\, G(\tfrac{1}{x},\tfrac{1}{y},t)\right]$$

$$= xy - \tfrac{1}{x}y - y\tfrac{1}{x} + \tfrac{1}{xy} = \left(x-\tfrac{1}{x}\right)\left(y-\tfrac{1}{y}\right)$$

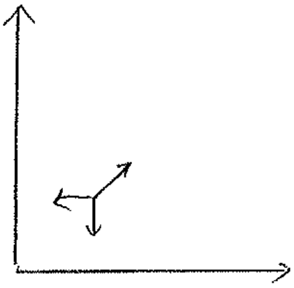
so that $$G(x,y,t) = \frac{1}{xy}\left[\frac{(x-\frac{1}{x})(y-\frac{1}{y})}{K(x,y,t)}\right]_{\geq (x,y)}$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{(z-\frac{1}{z})(w-\frac{1}{w})}{K(z,w,t)}\, \frac{dz}{z(z-x)}\, \frac{dw}{w(w-y)}$$

<u>exercise:</u> show that

$$G(x,0,t) = \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{(z-\frac{1}{z})(w-\frac{1}{w})}{K(z,w,t)}\, \frac{dz}{z(z-x)}\, \frac{dw}{w^2}$$

reduces to the previous result



$$\underbrace{\left[1-t\left(\frac{1}{x}+\frac{1}{y}+xy\right)\right]}_{K(x,y,t)} xy\, G(x,y,t) = xy - tx\, G(x,0,t) - ty\, G(0,y,t)$$

$$= xy - R(x) - R(y)$$

[$x \leftrightarrow y$ symmetry]

Symmetry: $K(x,y,t) = K\left(\frac{1}{xy}, y, t\right) = K\left(\frac{1}{xy}, x, t\right) = \ldots$

$$\begin{tikzcd}
(x,y) \arrow[r] & \left(\frac{1}{xy}, y\right) \arrow[r] & \left(\frac{1}{xy}, x\right) \arrow[d] \\
\left(x, \frac{1}{xy}\right) \arrow[u] & \left(y, \frac{1}{xy}\right) \arrow[l] & (y, x) \arrow[l]
\end{tikzcd}$$

C_6 cycle

$$y^2 - \frac{1}{x}\left(\frac{1}{t}-\frac{1}{x}\right)y + \frac{1}{x} = 0 \qquad Y_0+Y_1 = \frac{1}{x}\left(\frac{1}{t}-\frac{1}{x}\right),\ Y_0 Y_1 = \frac{1}{x}$$

$$\leadsto\ Y_0(x,t) = \frac{1}{2t}\left[\frac{1}{x}\left(1-\frac{t}{x}\right) \mp \sqrt{\frac{1}{x}\left(1-\frac{t}{x}\right)^2 - \frac{4}{x}}\right]$$

$$Y_0(x,t) = t + \frac{1}{x}t^2 + O(t^3), \quad Y_1(x,t) = \frac{1}{xt} - \frac{1}{x^2} - t - \frac{1}{x}t^2 + O(t^3)$$

Y

in iteration cycle, pick $y = Y_0(x,t)$:

$$\begin{tikzcd}
(x, Y_0) \arrow[r] & (Y_1, Y_0) \arrow[r] & (Y_1, x) \arrow[d] \\
(x, Y_1) \arrow[u] & (Y_0, Y_1) \arrow[l] & (Y_0, x) \arrow[l]
\end{tikzcd}$$

$$\rightsquigarrow \quad R(x) + R(Y_0) = xY_0 \sim xt$$

$$R(Y_1) + R(Y_0) = Y_0 Y_1 = \frac{1}{x}$$

$$\left[R(Y_1) + R(x) = Y_1 x = \frac{1}{Y_0} \sim \frac{1}{t} \qquad \text{no good} \right]$$

rewrite:
$$\underline{R(Y_0) - xY_0 = -R(x)}$$

$$\underline{R(Y_1) - xY_1} = Y_0 Y_1 - \underbrace{R(Y_0) - xY_1}$$

$$Y_0 + Y_1 = \frac{1}{x}\left(\frac{1}{t} - \frac{1}{x}\right)$$

$$= \frac{1}{x} + R(x) - xY_0 - xY_1$$

$$= \frac{1}{x} + R(x) - \left(\frac{1}{t} - \frac{1}{x}\right)$$

$$= \underline{R(x) + \frac{2}{x} - \frac{1}{t}}$$

so that
$$\frac{R(Y_0) - R(Y_1)}{Y_0 - Y_1} - x = tx \, \frac{2R(x) + \frac{2}{x} - \frac{1}{t}}{\sqrt{\delta(x,t)}}$$

where $\delta(x,t) = \left(1 - t\frac{1}{x}\right)^2 - 4t^2 x$. Use the factorisation lemma to argue that

$$\delta(x,t) = D(t)\, \Delta(x,t)\, \bar{\Delta}\left(\tfrac{1}{x}, t\right)$$

$$\left[\begin{array}{ll} \text{three roots } X_0, X_1, X_2 : & D(t) = 4t^2 X_2, \quad \Delta(x,t) = 1 - \frac{x}{X_2} \\ X_{0,1} \sim t \quad X_2 \sim \frac{1}{t^2} & \bar{\Delta}\left(\frac{1}{x}, t\right) = \left(1 - \frac{X_0}{x}\right)\left(1 - \frac{X_1}{x}\right) \end{array} \right]$$

therefore

$$\sqrt{\bar{\Delta}\left(\frac{1}{x}, t\right)} \left[\frac{R(y_0) - R(y_1)}{y_0 - y_1} - x\right] = t \, \frac{2xR(x) + 2 - \frac{x}{t}}{\sqrt{D(t)\,\Delta(x,t)}}$$

extracting the positive part gives

$$-x = \frac{t}{\sqrt{D(t)}} \left[\frac{2xR(x) + 2 - \frac{x}{t}}{\sqrt{\Delta(x,t)}} - 2\right]$$

rewriting (exercise) gives

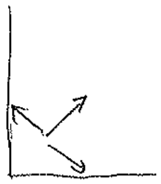
<u>Theorem</u>

$$G(x,0,t) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x}\right)\sqrt{1 - xW^2}\right)$$

where $W = t(2 + W^3)$ defines the power series $W = W(t)$

some further work gives for walks returning to the origin

$$c_{3n} = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n}$$



$$\left[1 - t\left(\frac{x}{y} + \frac{y}{x} + xy\right)\right] xy\, G(x,y,t) = xy - t x^2 G(x,0,t) - t y^2 G(0,y,t)$$

$K(x,y,t)=0 \rightsquigarrow \frac{1}{y^2} - \frac{1}{tx}\frac{1}{y} + \left(\frac{1}{x^2}+1\right) = 0$

$\rightsquigarrow$ $y_{0,1}$ satisfy $\frac{1}{y_0} + \frac{1}{y_1} = \frac{1}{t}\frac{1}{x}$ $\rightsquigarrow$ 3-term recurrence

iteration: $K(x_n, x_{n+1}, t) = 0$ gives

$\ldots, x_{-1} = y_0, \; x_0 = x, \; x_1 = y_1, \; x_2 = \ldots$ with x_n given by

$$\frac{1}{x_n} = \alpha \lambda^n + \beta \lambda^{-n}, \quad \lambda + \frac{1}{\lambda} = \frac{1}{t}, \quad \alpha + \beta = \frac{1}{x}, \; \alpha\lambda + \frac{\beta}{\lambda} = \frac{1}{y_1}$$

<u>exercise</u>: check that $x_n = x t^n + O(t^{n+1})$

$K(x_n, x_{n+1}, t) = 0 \rightsquigarrow t x_n^2 G(x_n, 0, t) = x_n x_{n+1} - t x_{n+1}^2 G(x_{n+1}, 0, t)$

leads to
$$G(x,0,t) = \frac{1}{x^2 t} \sum_{k=0}^{\infty} (-1)^k x_k x_{k+1}$$

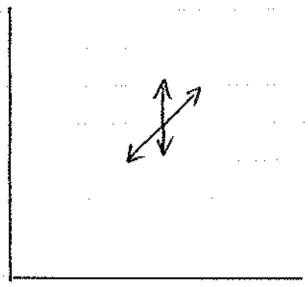
$$G(1,1,t) = \frac{1 - 2t\, G(1,0,t)}{1 - 3t}$$

exercise: $c_n \sim \left(1 - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{F_{2k-1} F_{2k+1}}\right) 3^n$

[Note: $G(x,y,t)$ is not differentiably finite (poles accumulate)]

<u>Conjecture</u> Symmetries of the Kernel determine whether the generating function is differentiably finite

<u>Example: Gessel walks</u>



$$K(x,y,t)\,G(x,y,t) = 1-\left(\frac{1}{xy}+\frac{1}{y}\right)G(x,0,t) - \frac{1}{xy}\,G(0,y,t) + \frac{1}{xy}\,G(0,0,t)$$

$$K(x,y,t) = 1-t\left(xy+y+\frac{1}{y}+\frac{1}{xy}\right)$$

iteration generates C_8:

$$\begin{tikzcd}
(x,y) \arrow[r] & \left(x,\frac{1}{xy}\right) \arrow[r] & \left(xy^2,\frac{1}{xy}\right) \arrow[r] & \left(xy^2,\frac{1}{y}\right) \arrow[d] \\
\left(\frac{1}{xy^2},y\right) \arrow[u] & \left(\frac{1}{xy^2},xy\right) \arrow[l] & \left(\frac{1}{x},xy\right) \arrow[l] & \left(\frac{1}{x},\frac{1}{y}\right) \arrow[l]
\end{tikzcd}$$

Kernel method fails (so far).

<u>Theorem</u>

$$C_{2n} = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} \qquad (a)_n = a(a+1)\dots(a+n-1)$$

Conjecture by Gessel 2001, Proof by Kauers, Koutschan and Zeilberger, 2008