Area-perimeter generating functions of lattice walks: $q$-series and their asymptotics

(A lattice model of vesicles attached to a skewed surface)

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Motivation
- Vesicle Generating Function
- Singularity Diagram
- Scaling Function

From Lattice Walks to Basic Hypergeometric Series
- $q$-Deformed Algebraic Equations
- $q$-Difference Equations
- Basic Hypergeometric Series

Asymptotic Analysis
- Contour Integral Representation
- Saddle Point Analysis
- Uniform Asymptotics

Outlook
Outline

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4 Outlook
Vesicle Generating Function

- 3-dim vesicle (bubble) with surface and volume
- 2-dim lattice model: polygons on the square lattice

$c_{m,n}$ number of polygons with area $m$ and perimeter $2n$
Vesicle Generating Function

- 3-dim vesicle (bubble) with surface and volume
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\( c_{m,n} \) number of polygons with area \( m \) and perimeter \( 2n \)

\[ G(x, q) = \sum_{n,m} c_{m,n} x^n q^m \]

Wanted:
- an explicit formula for \( G(x, q) \)
- singularity structure, e.g. \( q_c(x) \)
Folklore: universal behaviour near a “critical point”
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\[ G^{\text{sing}}(x, q) \sim (1 - q)^{-\gamma t} f \left( [1 - q]^{-\phi} [x_t - x] \right) \]

as \( q \to 1 \) and \( x \to x_t \) with 
\[ z = [1 - q]^{-\phi} [x_t - x] \] fixed.
**Scaling Function**

*Surprisingly often* \( f(z) = -\text{Ai}'(z)/\text{Ai}(z) \)
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- Square lattice vesicle generating function
  - Enumeration of \( c_{m,n} \), numerical analysis of moments (Richard, Guttmann, Jensen)
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  - Rigorous derivation (Prellberg)
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- Area statistics of outer boundary of random loops
  - Monte-Carlo simulation (Richard)
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- $q$-Analogue of the Painlevé II equation (Witte)
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4 Outlook
Example 1: Dyck Paths

\[ 2n = 14 \text{ steps enclosing an area of size } m = 9 \]
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\[ G(t, q) = \sum_{m,n} c_{m,n} t^n q^m \]

t counts pairs of up/down steps, q counts enclosed area
Example 1: Dyck Paths

A functional equation

\[ G(t, q) = 1 + tG(qt, q)G(t, q) \]
Example 1: Dyck Paths

- A functional equation
  \[ G(t, q) = 1 + tG(qt, q)G(t, q) \]

- \( C(t) = G(t, 1) \) satisfies \( C(t) = 1 + tC(t)^2 \)
  \[ C(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n=0}^{\infty} \frac{t^n}{n + 1} \binom{2n}{n} \]

Generating function of Catalan numbers
Example 2: A Pair of Directed Walks

Two directed walks not allowed to cross
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Two directed walks not allowed to cross

\[ G(x, y, q) = \sum_{m, n_x, n_y} c_{m, n_x, n_y} x^{n_x} y^{n_y} q^m \]

\( x \) and \( y \) count pairs of east and north steps, \( q \) counts enclosed area
Example 2: A Pair of Directed Walks

- A functional equation

\[ G(x, y, q) = 1 + yG(qx, y, q) + xG(x, y, q) + yG(qx, y, q)xG(x, y, q) \]
Example 2: A Pair of Directed Walks

- A functional equation

\[ G(x, y, q) = 1 + yG(qx, y, q) + xG(x, y, q) + yG(qx, y, q)xG(x, y, q) \]

- \( G(t, t, 1) = 1 + tC(t) \) Catalan generating function
Example 3: Partially Directed Walks Above $y = x$

- A functional equation

\[ G(x, y, q) = 1 + yG(qx, y, q)G(x, y, q) + y(G(qx, y, q) - 1)y \]
Example 3: Partially Directed Walks Above $y = x$

- A functional equation

\[
G(x, y, q) = 1 + yG(qx, y, q)G(x, y, q) + y(G(qx, y, q) - 1)y
\]

- $G(x, y, 1) = C \left( \frac{xy}{1-y^2} \right)$ Catalan generating function
Summary of the Examples

Different $q$-deformations of Catalan-type generating functions:

- Dyck paths
  \[ G(t) = 1 + tG(t)G(qt) \]
- Pair of directed walks
  \[ G(x) = (1 + xG(x))(1 + yG(qx)) \]
- Partially directed walks above the diagonal
  \[ G(x) = 1 + xyG(x)G(qx) + y^2(G(qx) - 1) \]
Example 1: Solving $G(t) = 1 + tG(t)G(qt)$

An aside:
- $G(t)$ admits a nice continued fraction expansion

$$G(t) = \frac{1}{1 - \frac{t}{1 - \frac{qt}{1 - \frac{q^2t}{1 - \ldots}}}}$$

- Connections with orthogonal polynomials, combinatorics of weighted lattice paths, ...
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- Connections with orthogonal polynomials, combinatorics of weighted lattice paths, ...
- However, useless for finer asymptotic analysis of $q \to 1$. 

Thomas Prellberg
INI Workshop on Discrete Systems and Special Functions
Example 1: Solving $G(t) = 1 + tG(t)G(qt)$

Better:

- Linearise the functional equation using

$$G(t) = \frac{H(qt)}{H(t)}$$
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$$H(qt) = H(t) + tH(q^2t)$$
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- Explicit solution
  $$H(t) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q; q)_n} = \phi_1(-; 0; q, -t)$$
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$[\phi_1(-; 0; q, -qt)$ a $q$-Airy function (Ismail)]
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Better:
- Linearise the functional equation using
  \[
  G(x) = \frac{1}{x} \left( \frac{H(qx)}{H(x)} - 1 \right)
  \]
- Obtain a linear $q$-difference equation
  \[
  q(H(qx) - H(x)) = qxH(qx) + y(H(q^2x) - H(qx))
  \]
- Explicit solution

\[
H(t) = \sum_{n=0}^{\infty} \frac{q^\binom{n}{2}(-x)^n}{(y; q)_n(q; q)_n} = _1\phi_1(0; y; q, x)
\]
Example 3: \( G(x) = 1 + xyG(x)G(qx) + y^2(G(qx) - 1) \)

Better:

- Linearise the functional equation using
  \[
  G(x) = \frac{y}{x} \left( \frac{H(qx)}{H(x)} - 1 \right)
  \]

- Obtain a linear \( q \)-difference equation
  \[
  q(H(qx) - H(x)) = qx \left( \frac{1}{y} - y \right) H(qx) + y^2 \left( H(q^2x) - H(qx) \right)
  \]

- Explicit solution
  \[
  H(t) = \sum_{n=0}^{\infty} \frac{(-x(1 - y^2)/y)^n}{(y^2; q)_n(q; q)_n} = {}_2\phi_1(0, 0; y^2; q, -x(1 - y^2)/y)
  \]
Summary:

Different $q$-deformations of Catalan-type generating functions:

- **Dyck paths**
  
  $$ G(t, q) = \frac{0\phi_1(0; q, -qt)}{0\phi_1(0; q, -t)} $$

- **Pair of directed walks**
  
  $$ G(x, y, q) = \frac{1}{x} \left( \frac{1\phi_1(0; y, q, qx)}{1\phi_1(0; y, q, x)} - 1 \right) $$

- **Partially directed walks above the diagonal**
  
  $$ G(x, y, q) = \frac{y}{x} \left( \frac{2\phi_1(0, 0; y^2, q, qx(y - 1/y))}{2\phi_1(0, 0; y^2, q, x(y - 1/y))} - 1 \right) $$
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4 Outlook
The full generating function is a quotient of $q$-series, e.g.

$$G(t, q) = \sum_{n=0}^{\infty} \frac{q^n (-t)^n}{(q; q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^n (-t)^n}{(q; q)_n}$$
A Puzzle

- The full generating function is a quotient of $q$-series, e.g.

$$G(t, q) = \frac{\sum_{n=0}^{\infty} q^{n^2} (-t)^n}{\sum_{n=0}^{\infty} q^{n^2+n} (-t)^n} \frac{(q; q)_n}{(q; q)_n}$$

- However, for $q = 1$ we have a simple algebraic generating function

$$G(t, 1) = \frac{1 - \sqrt{1 - 4t}}{2t}$$
A Puzzle

- The full generating function is a quotient of \(q\)-series, e.g.

\[
G(t, q) = \frac{\sum_{n=0}^{\infty} q^{n^2} (-t)^n}{\sum_{n=0}^{\infty} q^{n^2-n} (-t)^n} \frac{(q; q)_{\infty}}{(q; q)_{\infty}}
\]

- However, for \(q = 1\) we have a simple algebraic generating function

\[
G(t, 1) = \frac{1 - \sqrt{1 - 4t}}{2t}
\]

*How can one understand the limit \(q \rightarrow 1\)?*
A Standard Trick For Evaluating Alternating Series

- Write an alternating series as a contour integral

\[ \sum_{n=0}^{\infty} (-x)^n c_n = \frac{1}{2\pi i} \oint_C x^s c(s) \frac{\pi}{\sin(\pi s)} \, ds \]

\( C \) runs counterclockwise around the zeros of \( \sin(\pi s) \)
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- For example,

\[ \exp(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \frac{1}{2\pi i} \int_{-c-i\infty}^{c+i\infty} x^s \Gamma(-s) ds \]

where \( c > 0 \) (here, we have used \( \Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s) \))
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where \( c > 0 \) (here, we have used \( \Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s) \))

Find suitable q-version for this trick
Use that

$$ \text{Res } [(z; q)^{-1}; z = q^{-n}] = -\frac{(-1)^n q^{\frac{n}{2}}}{(q; q)_n (q; q)_\infty} \quad n = 0, 1, 2, \ldots $$
Contour Integral Representation

Use that

$$\text{Res } [(z; q)_{\infty}^{-1}; z = q^{-n}] = -\frac{(-1)^n q^n}{(q; q)_n (q; q)_{\infty}}$$

$$n = 0, 1, 2, \ldots$$

to prove that

**Lemma**

*For complex t with $|\arg(x)| < \pi$, non-negative integer n, and $0 < q < 1$ we have for $0 < \rho < 1$*

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q; q)_n} = \frac{(q; q)_{\infty}}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} \frac{z^{1/2} \log q}{(z; q)_{\infty}} \frac{z - \log t}{\sqrt{z}} dz$$
Some Asymptotics

Approximate $\log(z; q)_{\infty} \sim \frac{1}{\log q} \operatorname{Li}_2(z) + \frac{1}{2} \log(1 - z)$ to get

**Lemma**

For $0 < t < 1$ and with $\varepsilon = -\log q$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q; q)_n} = \frac{(q; q)_{\infty}}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} e^{\frac{1}{\varepsilon} \left[ -\frac{1}{2} (\log z)^2 + \log(z) \log(t) + \operatorname{Li}_2(z) \right]} \sqrt{\frac{z}{1 - z}} \, dz \left[ 1 + O(\varepsilon) \right]$$

where $t < \rho < 1$
Some Asymptotics

Approximate \( \log(z; q) \infty \sim \frac{1}{\log q} \text{Li}_2(z) + \frac{1}{2} \log(1 - z) \) to get

Lemma

For \( 0 < t < 1 \) and with \( \varepsilon = -\log q \)

\[
\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(q; q)_n} =
(q; q)_\infty \int_{-\rho-i\infty}^{\rho+i\infty} e^{\frac{1}{\varepsilon} \left[ -\frac{1}{2} (\log z)^2 + \log(z) \log(t) + \text{Li}_2(z) \right]} \sqrt{\frac{z}{1 - z}} \, dz \left[ 1 + O(\varepsilon) \right]
\]

where \( t < \rho < 1 \)

We find a Laplace-type integral, where the saddles are given by

\[
0 = \frac{d}{dz} \left[ -\frac{1}{2} (\log z)^2 + \log(z) \log(t) + \text{Li}_2(z) \right]
\]
Saddle Point Analysis

- The asymptotics of
  \[ \int_{C} e^{\frac{1}{\epsilon} g(z)} f(z) dz \]
  is dominated by the saddles with \( g'(z) = 0 \).
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For \( g(z) = -\frac{1}{2} (\log z)^2 + \log(z) \log(t) + \text{Li}_2(z) \) we find two saddles given by the zeros of

\[ z(1 - z) = t \quad \Rightarrow \quad z = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4t} \]
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As \( t \) approaches \( t_t = 1/4 \), the saddles coalesce
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As \( t \) approaches \( t_t = 1/4 \), the saddles coalesce

Standard procedure: reparametrise locally by a cubic and compute a uniform asymptotic expansion (involving Airy functions)…
Saddle Point Summary:

Saddle Point coalescence occurs in all three cases:

- Dyck paths, $\phi_1(-; 0; q, -t)$:

$$g(z) = -\frac{1}{2} (\log z)^2 + \log(z) \log(t) + \text{Li}_2(z) \Rightarrow (z - 1)z + t = 0$$
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Saddle Point coalescence occurs in all three cases:

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  \]

- **Pair of directed walks, \( _1\phi_1(0; y; q, x) \):**
  \[
  g(z) = -\text{Li}_2\left(\frac{y}{z}\right) + \log(z) \log(x) + \text{Li}_2(z) \quad \Rightarrow \quad (z - 1)(z - y) + zx = 0
  \]
Saddle Point Summary:

Saddle Point coalescence occurs in all three cases:

- **Dyck paths,** $0\phi_1(-; 0; q, -t)$:
  \[
g(z) = -\frac{1}{2} (\log z)^2 + \log(z) \log(t) + \text{Li}_2(z) \quad \Rightarrow \quad (z - 1)z + t = 0
\]

- **Pair of directed walks,** $1\phi_1(0; y; q, x)$:
  \[
g(z) = -\text{Li}_2(y/z) + \log(z) \log(x) + \text{Li}_2(z) \quad \Rightarrow \quad (z - 1)(z - y) + zx = 0
\]

- **Part. directed walks above the diagonal,** $2\phi_1(0, 0; y^2; q, x(y - 1/y))$:
  \[
g(z) = \ldots \quad \Rightarrow \quad (z - 1)(z - y^2) + z^2x(1/y - y) = 0
\]
Uniform Asymptotics

Theorem

Let $0 < t < 1$ and $\varepsilon = -\log q$. Then, as $\varepsilon \to 0^+$,

$$G(t, q) \sim \frac{1}{2} \left(1 - \sqrt{1 - 4t}\right) \left[-\frac{\text{Ai}'(\alpha \varepsilon^{-2/3})}{\alpha^{1/2} \varepsilon^{-1/3} \text{Ai}(\alpha \varepsilon^{-2/3})}\right]$$

where $\alpha = \alpha(t)$ is an explicitly given function of $t$. In particular,

$$\alpha(t) \sim 1 - 4t \quad \text{as } t \to 1/4$$
**Theorem**

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where $\alpha = \alpha(t)$ is an explicitly given function of $t$. In particular,

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Some remarks:

- Uniform convergence to $G(t, 1) = \frac{1}{2} \left( 1 - \sqrt{1 - 4t} \right)$
Uniform Asymptotics

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Let $0 < t < 1$ and $\varepsilon = -\log q$. Then, as $\varepsilon \to 0^+$,

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Some remarks:

- Uniform convergence to $G(t, 1) = \frac{1}{2} \left( 1 - \sqrt{1 - 4t} \right)$
- Scaling function $f(z) = -\text{Ai}'(z)/\text{Ai}(z)$
Theorem

Let $0 < t < 1$ and $\varepsilon = -\log q$. Then, as $\varepsilon \to 0^+$,

$$G(t, q) \sim \frac{1}{2} \left(1 - \sqrt{1 - 4t} \left[-\frac{\text{Ai}'(\alpha \varepsilon^{-2/3})}{\alpha^{1/2} \varepsilon^{-1/3} \text{Ai}(\alpha \varepsilon^{-2/3})}\right]\right)$$

where $\alpha = \alpha(t)$ is an explicitly given function of $t$. In particular,

$$\alpha(t) \sim 1 - 4t \quad \text{as } t \to 1/4$$

Some remarks:

- Uniform convergence to $G(t, 1) = \frac{1}{2} \left(1 - \sqrt{1 - 4t}\right)$
- Scaling function $f(z) = -\text{Ai}'(z)/\text{Ai}(z)$
- Stronger than scaling limit which keeps $z = (1 - 4t)\varepsilon^{-2/3}$ fixed
Theorem

Let $0 < t < 1$ and $\varepsilon = -\log q$. Then, as $\varepsilon \to 0^+$,

$$G(t, q) \sim \frac{1}{2} \left( 1 - \sqrt{1 - 4t} \left[ -\frac{\text{Ai}'(\alpha \varepsilon^{-2/3})}{\alpha^{1/2} \varepsilon^{-1/3} \text{Ai}(\alpha \varepsilon^{-2/3})} \right] \right)$$

where $\alpha = \alpha(t)$ is an explicitly given function of $t$. In particular,

$$\alpha(t) \sim 1 - 4t \quad \text{as } t \to 1/4$$

Some remarks:

- Uniform convergence to $G(t, 1) = \frac{1}{2} \left( 1 - \sqrt{1 - 4t} \right)$
- Scaling function $f(z) = -\text{Ai}'(z)/\text{Ai}(z)$
- Stronger than scaling limit which keeps $z = (1 - 4t)\varepsilon^{-2/3}$ fixed

The result is completely analogous for the other examples.
Outline

1 Motivation
   - Vesicle Generating Function
   - Singularity Diagram
   - Scaling Function

2 From Lattice Walks to Basic Hypergeometric Series
   - $q$-Deformed Algebraic Equations
   - $q$-Difference Equations
   - Basic Hypergeometric Series

3 Asymptotic Analysis
   - Contour Integral Representation
   - Saddle Point Analysis
   - Uniform Asymptotics

4 Outlook
Outlook

So far:

- simple $q$-algebraic equation
- simple $q$-series solution
- contour integral
- saddle-point analysis
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The End