On the Number of Walks in a Triangular Domain

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Topic Outline

1. Walks in a Triangle
2. Functional Equations
3. The Kernel Method
4. Bijections?
5. Summary and Open Problem
Outline

1 Walks in a Triangle
2 Functional Equations
3 The Kernel Method
4 Bijections?
5 Summary and Open Problem
Walks in a Triangle

Walk with 10 steps inside a triangle of side-length 6
Counting Walks

Parameters

- Side-length $L$
- Number of steps $n$
- Starting point $a$
- End point $b$
Counting Walks

Parameters
- Side-length $L$
- Number of steps $n$
- Starting point $a$
- End point $b$

Number of $n$-step walks from $a$ to $b$ within triangle of side-length $L$

$$c_{n,L}^{a,b}$$

Generating function

$$Z_{L}^{a,b}(t) = \sum c_{n,L}^{a,b} t^{n}$$
Counting Walks

Generating function

\[ Z_{L}^{a,b}(t) = \sum c_{n,L}^{a,b} t^{n} \]
Counting Walks

Generating function

$$Z_{L}^{a,b}(t) = \sum c_{n,L}^{a,b} t^{n}$$

Finite transition matrix ⇒ Rational generating function

$$\binom{L+2}{2}$$ vertices ⇒ Degree of polynomials grows quadratically in $L$
Counting Walks

Generating function

\[ Z_{L}^{a,b}(t) = \sum c_{n,L}^{a,b} t^n \]

Finite transition matrix \implies\ Rational generating function

\[ \binom{L+2}{2} \text{ vertices} \implies\ \text{Degree of polynomials grows quadratically in } L \]

It is surprisingly difficult to give a closed-form expression for \( Z_{L}^{a,b}(t) \)
Some Notation

Associate to each vertex three coordinates \((n_x, n_y, n_z)\)

\[
\begin{align*}
n_x + n_y + n_z &= L \\
\end{align*}
\]

For example, the point in the triangle above is given by \(p = (1, 2, 3)\)
Consider new generating function by summing over end-point positions

\[
G_L^a(x, y, z; t) = \sum_{n_x, n_y, n_z} Z_L^{a, (n_x, n_y, n_z)}(t)x^{n_x}y^{n_y}z^{n_z}
\]
Some Notation

Fix $L$ and $a$, drop $t$ and write

$$G(x, y, z) \equiv G^a_L(x, y, z; t)$$

As $n_x + n_y + n_z = L$, $G$ is homogeneous of degree $L$

$$G(\gamma x, \gamma y, \gamma z) = \gamma^L G(x, y, z)$$
Changing the Dimension

Triangle of side-length $L$, starting point $a = (u, v, w)$ with $u + v + w = L$:

$$G(x, y, z) = \sum_{n_x, n_y, n_z, t} C_{n,L}^{(u,v,w),(n_x,n_y,n_z)} x^{n_x} y^{n_y} z^{n_z} t^n$$
**Changing the Dimension**

Triangle of side-length $L$, starting point $a = (u, v, w)$ with $u + v + w = L$:

$$G(x, y, z) = \sum_{n_x, n_y, n_z, t} c_{n,L}^{(u,v,w),(n_x,n_y,n_z)} x^{n_x} y^{n_y} z^{n_z} t^n$$

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$
Changing the Dimension

Triangle of side-length $L$, starting point $a = (u, v, w)$ with $u + v + w = L$:

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Obvious generalisation to tetrahedron and higher dimensions
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   - The Line
   - The Triangle

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5. Summary and Open Problem
The Line

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$
The Line

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v), (n_x, n_y)} x^n x y^n t^n$$

$$G(x, y) = x^u y^v$$

zero-length walk
Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$

$$G(x, y) = x^u y^v$$

zero-length walk

$$+ G(x, y) t \frac{x}{y}$$

take walk and add a step to the right
The Line

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n,L}^{(u,v),(n_x,n_y)} x^{n_x} y^{n_y} t^n$$

\[
\begin{align*}
G(x, y) &= x^u y^v \quad \text{zero-length walk} \\
&+ G(x, y) t \frac{x}{y} \quad \text{take walk and add a step to the right} \\
&+ G(x, y) t \frac{y}{x} \quad \text{take walk and add a step to the left}
\end{align*}
\]
Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n, L}^{(u, v), (n_x, n_y)} x^{n_x} y^{n_y} t^n$$

$$G(x, y) = x^u y^v$$  

zero-length walk

$$+ G(x, y) t^{x/y}$$  
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$$+ G(x, y) t^{y/x}$$  
take walk and add a step to the left

$$- G(x, 0) t^{x/y}$$  
forbid stepping past the right boundary
Line of length $L$, starting point $\mathbf{a} = (u, v)$ with $u + v = L$:

$$G(x, y) = \sum_{n_x, n_y, t} c_{n_x, n_y, t}^{(u, v)} x^{n_x} y^{n_y} t^n$$

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$$+ G(x, y) t \frac{y}{x}$$  take walk and add a step to the left

$$- G(x, 0) t \frac{x}{y}$$  forbid stepping past the right boundary

$$- G(0, y) t \frac{y}{x}$$  forbid stepping past the left boundary
The Line

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

Functional equation

\[
\left[1 - t \left( \frac{x}{y} + \frac{y}{x} \right) \right] G(x, y) = x^u y^v - t \frac{x}{y} G(x, 0) - t \frac{y}{x} G(0, y)
\]

Kernel
The Line

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

Functional equation

$$
\left[ 1 - t \left( \frac{x}{y} + \frac{y}{x} \right) \right] G(x, y) = x^u y^v - t \frac{x}{y} G(x, 0) - t \frac{y}{x} G(0, y)
$$

Note that the length $L$ only enters the functional equation through $x^u y^v$
The Line

Line of length $L$, starting point $a = (u, v)$ with $u + v = L$:

Functional equation

$$\left[1 - t \left(\frac{x}{y} + \frac{y}{x}\right)\right] G(x, y) = x^u y^v - t \frac{x}{y} G(x, 0) - t \frac{y}{x} G(0, y)$$

Note that the length $L$ only enters the functional equation through $x^u y^v$

The Kernel

$$K(x, y) = 1 - t \left(\frac{x}{y} + \frac{y}{x}\right)$$

will be central to finding a solution of this functional equation.
The Triangle

Triangle of side-length $L$, starting point $\mathbf{a} = (u, v, w)$ with $u + v + w = L$:
The Triangle

Triangle of side-length $L$, starting point $a = (u, v, w)$ with $u + v + w = L$:

\[
K(x, y, z)G(x, y, z) = x^u y^v z^w
- t \left( \frac{y}{x} + \frac{z}{x} \right) G(0, y, z)
- t \left( \frac{z}{y} + \frac{x}{y} \right) G(x, 0, z)
- t \left( \frac{x}{z} + \frac{y}{z} \right) G(x, y, 0)
\]

with the Kernel

\[
K(x, y, z) = 1 - t \left( \frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{x} \right)
\]
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   - Generalities
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The Kernel Method

Loosely speaking, the Kernel Method consists of

- setting the Kernel equal to zero and manipulating the RHS, or
- using variable transformations that leave the Kernel invariant and cancelling terms on the RHS, or
- staring at the stuff until you get a good idea . . .
The Kernel Method

Loosely speaking, the Kernel Method consists of

- setting the Kernel equal to zero and manipulating the RHS, or
- using variable transformations that leave the Kernel invariant and cancelling terms on the RHS, or
- staring at the stuff until you get a good idea . . .

This gives rise to various variations, known as the elementary Kernel method, the algebraic Kernel method, the obstinate Kernel method, the iterative Kernel method, and so forth.

The experts will recognize in what follows yet another variation of the theme.
Recall the functional equation

\[ K(x, y)G(x, y) = x^u y^v - t \frac{x}{y} G(x, 0) - t \frac{y}{x} G(0, y) \]

with the Kernel

\[ K(x, y) = 1 - t \left( \frac{x}{y} + \frac{y}{x} \right) \]
Recall the functional equation

\[ K(x, y)G(x, y) = x^u y^v - t \frac{x}{y} G(x, 0) - t \frac{y}{x} G(0, y) \]

with the Kernel

\[ K(x, y) = 1 - t \left( \frac{x}{y} + \frac{y}{x} \right) \]

The Kernel is invariant under

\[ K(x, y) = K(y, x) = K(\lambda x, \lambda y) \]

In particular,

\[ K(p, 1) = K(1, p) = 1 - t(p + 1/p) \]
Now choose $p$ such that

$$K(p, 1) = K(1, p) = 1 - t(p + 1/p) = 0$$
The Line

Now choose $p$ such that

$$K(p, 1) = K(1, p) = 1 - t(p + 1/p) = 0$$

- Specialize $(x, y) = (1, p)$ and $(x, y) = (p, 1)$

$$0 = p^u - tpG(p, 0) - \frac{t}{p}G(0, 1)$$

$$0 = p^v - \frac{t}{p}G(1, 0) - tpG(0, p)$$
Now choose $p$ such that

$$K(p, 1) = K(1, p) = 1 - t(p + 1/p) = 0$$

- Specialize $(x, y) = (1, p)$ and $(x, y) = (p, 1)$

  $$0 = p^u - tpG(p, 0) - \frac{t}{p}G(0, 1)$$

  $$0 = p^v - \frac{t}{p}G(1, 0) - tpG(0, p)$$

- Use homogeneity $G(p, 0) = p^L G(1, 0)$ and $G(0, p) = p^L G(0, 1)$

  $$0 = p^u - tp^{1+L} G(1, 0) - \frac{t}{p}G(0, 1)$$

  $$0 = p^v - \frac{t}{p}G(1, 0) - tp^{1+L} G(0, 1)$$
The Line

This leads to a complete solution

**Proposition**

The generating function $G(x, y)$ counting $n$-step walks on a line of length $L$ starting at $(u, v)$ is given by

$$G(x, y) = \frac{1}{1 - \frac{x}{y} + \frac{y}{x}} \left( x^u y^v - \frac{x^{u+v+1} p^{v+1}(1 - p^{2u+2})}{y(1 - p^{2u+2v+4})} - \frac{y^{u+v+1} p^{u+1}(1 - p^{2v+2})}{x(1 - p^{2u+2v+4})} \right),$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t}.$$
Specializing to $x = y = 1$, the result looks quite pleasant

**Corollary**

The generating function $G(1, 1)$ counting $n$-step walks on a line of length $L$ starting at $(u, v)$ with no restrictions on the endpoint is given by

$$G(1, 1) = \frac{(1 + p^2)(1 - p^{u+1})(1 - p^{v+1})}{(1 - p)^2(1 + p^{u+v+2})},$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t}.$$
The Triangle

The Kernel

\[ K(x, y, z) = 1 - t \left( \frac{y}{x} + \frac{z}{x} + \frac{y}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{x} \right) \]

is invariant under

\[ K(1, 1, p) = K(1, p, 1) = K(p, 1, 1) = K(1, p, p) = K(p, 1, p) = K(p, p, 1) \]
The Triangle

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\[ K(x, y, z) = 1 - t \left( \frac{y}{x} + \frac{z}{x} + \frac{z}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{z} \right) \]

is invariant under

\[ K(1, 1, p) = K(1, p, 1) = K(p, 1, 1) = K(1, p, p) = K(p, 1, p) = K(p, p, 1) \]

Choose \( p \) such that

\[ K(1, 1, p) = 1 - 2t(p + 1 + 1/p) = 0 \]
We find

\[
\begin{align*}
\frac{2t}{p} G(0, 1, 1) + t(1 + p)G(p, 0, 1) + t(p + 1)G(p, 1, 0) &= p^u \\
\frac{2t}{p} G(1, 0, 1) + t(p + 1)G(0, p, 1) + t(1 + p)G(1, p, 0) &= p^v \\
\frac{2t}{p} G(1, 1, 0) + t(1 + p)G(0, 1, p) + t(p + 1)G(1, 0, p) &= p^w \\
2tpG(0, p, p) + t \left(1 + \frac{1}{p}\right) G(1, 0, p) + t \left(\frac{1}{p} + 1\right) G(1, p, 0) &= p^v p^w \\
2tpG(p, 0, p) + t \left(\frac{1}{p} + 1\right) G(0, 1, p) + t \left(1 + \frac{1}{p}\right) G(p, 1, 0) &= p^u p^w \\
2tpG(p, p, 0) + t \left(1 + \frac{1}{p}\right) G(0, p, 1) + t \left(\frac{1}{p} + 1\right) G(p, 0, 1) &= p^u p^v
\end{align*}
\]
While this is insufficient to find a full solution, we are able to show

**Theorem (Mortimer, Prellberg)**

The generating function $G(1, 1, 1)$ which counts $n$-step walks in a triangle of side-length $L$ starting at $(u, v, w)$ with no restrictions on the endpoint is given by

$$G(1, 1, 1) = \frac{(1 - p^3)(1 - p^{u+1})(1 - p^{v+1})(1 - p^{w+1})}{(1 - p)^3(1 - p^{u+v+w+3})},$$

where

$$p = \frac{1 - 2t - \sqrt{(1 + 2t)(1 - 6t)}}{4t}$$
Starting the walks in a corner of the triangle, we find

**Corollary (Mortimer, Prellberg)**

The generating function which counts \( n \)-step walks in a triangle of side-length \( L \) starting at a chosen corner with no restrictions on the endpoint is given by

\[
\frac{(1 - p^3)(1 - p^{1+L})}{(1 - p)(1 - p^{3+L})}
\]

where

\[
p = \frac{1 - 2t - \sqrt{(1 + 2t)(1 - 6t)}}{4t}
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$$

where

$$
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$$

It is not obvious that this is a rational function in $t$, but it is indeed one, and it has a very special structure . . .
Continued Fractions

For \( L = 2H \) even, the GF expands as a continued fraction of length \( H \),

\[
\frac{(1 - p^3)(1 - p^{1+2H})}{(1 - p)(1 - p^{3+2H})} = \frac{1}{1 - 2t - \frac{4t^2}{4t^2} \frac{1}{1 - 2t - \frac{4t^2}{4t^2} \ldots \frac{1}{1 - 2t - 4t^2}}} \quad \text{length } H
\]

and for \( L = 2H + 1 \) odd, a continued fraction of length \( H + 1 \),

\[
\frac{(1 - p^3)(1 - p^{2+2H})}{(1 - p)(1 - p^{4+2H})} = \frac{1}{1 - 2t - \frac{4t^2}{4t^2} \frac{1}{1 - 2t - \frac{4t^2}{4t^2} \ldots \frac{1}{1 - 2t}}} \quad \text{length } H + 1
\]
Bi-Colored Motzkin Paths

\begin{enumerate}
\item \(n\)-step walks starting in a corner of a triangle of side-length \(L = 2H + 1\) with arbitrary endpoint are in bijection with bi-colored \(n\)-step Motzkin paths in a strip of height \(H\).
\item \(n\)-step walks starting at a corner of a triangle of side-length \(L = 2H\) with arbitrary endpoint are in bijection with bi-colored \(n\)-step Motzkin paths in a strip of height \(H\), such that horizontal steps at height \(H\) are forbidden.
\end{enumerate}
**Corollary (Mortimer, Prellberg)**

(a) *n*-step walks starting in a corner of a triangle of side-length $L = 2H + 1$ with arbitrary endpoint are in bijection with bi-colored *n*-step Motzkin paths in a strip of height $H$.

(b) *n*-step walks starting at a corner of a triangle of side-length $L = 2H$ with arbitrary endpoint are in bijection with bi-colored *n*-step Motzkin paths in a strip of height $H$, such that horizontal steps at height $H$ are forbidden.
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- Set up general problem: walks on line, triangle, tetrahedron, ...
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- For a wedge ($L \to \infty$), there is a bijection via Young tableaus of height three (Eu, Adv Appl Math, 2010)
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Open Problem

Find a bijection for triangles!