

# The Combinatorics of the leading root of the partial theta function

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## Abstract

Recently Alan Sokal studied the leading root  $x_0(q)$  of the partial theta function  $\Theta_0(x, q) = \sum_{n=0}^{\infty} x^n q^{\binom{n}{2}}$ , considered as a formal power series. He proved that all the coefficients of

$$-x_0(q) = 1 + q + 2q^2 + 4q^3 + 9q^4 + \dots$$

are positive integers. I give here an explicit combinatorial interpretation of these coefficients. More precisely, I show that  $-x_0(q)$  enumerates rooted trees that are enriched by certain polyominoes, weighted according to their total area.

Alan Sokal proved in [8] that the partial theta function [1, 2]

$$\Theta_0(x, q) = \sum_{n=0}^{\infty} x^n q^{\binom{n}{2}} \tag{1}$$

admits a unique formal power series  $\xi_0(q) \in R[[q]]$  satisfying  $\Theta_0(-\xi_0(q), q) = 0$ , and that it has strictly positive integer coefficients,

$$\xi_0(q) = 1 + q + 2q^2 + 4q^3 + 9q^4 + 21q^5 + 52q^6 + 133q^7 + 351q^8 + 948q^9 + \dots \tag{2}$$

I present here a combinatorial interpretation of these coefficients in terms of rooted trees with vertices enriched with a certain class of objects related to polyominoes; I will define these objects more carefully below. The enrichment is such that the out-degree of the vertices in the tree is matched by a particular geometric parameter of the polyominoes. I will show that in fact these combinatorial interpretations are not unique, and can be extended to an uncountable family of different representations.

The partial theta function  $\Theta_0(x, q)$  can be seen as a special case of the three-variable Rogers-Ramanujan function

$$R(x, y, q) = \sum_{n=0}^{\infty} \frac{x^n y^{\binom{n}{2}}}{(1+q)(1+q+q^2)\dots(1+q+\dots+q^{n-1})}, \tag{3}$$

which seems to have intriguing structural properties, albeit many of them unproved at present [8]. Clearly  $\Theta_0(x, q) = R(x, q, 0)$ , and note also that  $R(x, q, q)$  has a particularly simple product representation,

$$R(x, q, q) = \prod_{n=0}^{\infty} (1 + x(1 - q)q^n), \quad (4)$$

whence one can write all roots explicitly as  $-x_n(q) = q^{-n} + q^{-n+1} + q^{-n+2} + \dots$  and the corresponding counting problem becomes trivial. It is an intriguing open problem to better characterise the roots of  $R(x, y, q)$  in general, and I present here the first step by providing a combinatorial interpretation of the leading root of  $R(x, q, 0)$ .

The combinatorial interpretation of the leading root  $x_0(q)$  of the partial theta function  $\Theta_0(x, q)$  relies on the following characterisation of roots of the partial theta function (1), where the standard notation  $(t; q)_n = \prod_{i=0}^{n-1} (1 - tq^i)$  and  $(t; q)_\infty = \prod_{i=0}^{\infty} (1 - tq^i)$  is used.

**Lemma 1** (Eqns. (3.2) and (4.9) in Sokal [8]). *The leading root  $x_0(q) = -\xi_0(q)$  of the partial theta function  $\Theta_0(x, q)$  satisfies*

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (\xi_0(q)q; q)_{n-1}} \quad (5)$$

and

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} \xi_0(q)^n}{(q; q)_n (\xi_0(q)q; q)_{n-1}}. \quad (6)$$

*Proof.* Using a result of Sokal [8, Lemma 2.1], the partial theta function (1) satisfies

$$\Theta_0(x, q) = (q; q)_\infty (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (-x; q)_n} \quad (7)$$

$$= (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} (-x)^n}{(q; q)_n (-x; q)_n} \quad (8)$$

as formal power series. From Eqn. (7) it follows that

$$\begin{aligned} \Theta_0(x, q) &= (q; q)_\infty (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n (-x; q)_n} \\ &= (q; q)_\infty (-xq; q)_\infty \left[ 1 + x + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (-xq; q)_{n-1}} \right]. \end{aligned} \quad (9)$$

Hence  $\Theta_0(-\xi_0(q), q) = 0$  implies that

$$0 = 1 - \xi_0(q) + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (\xi_0(q)q; q)_{n-1}}, \quad (10)$$

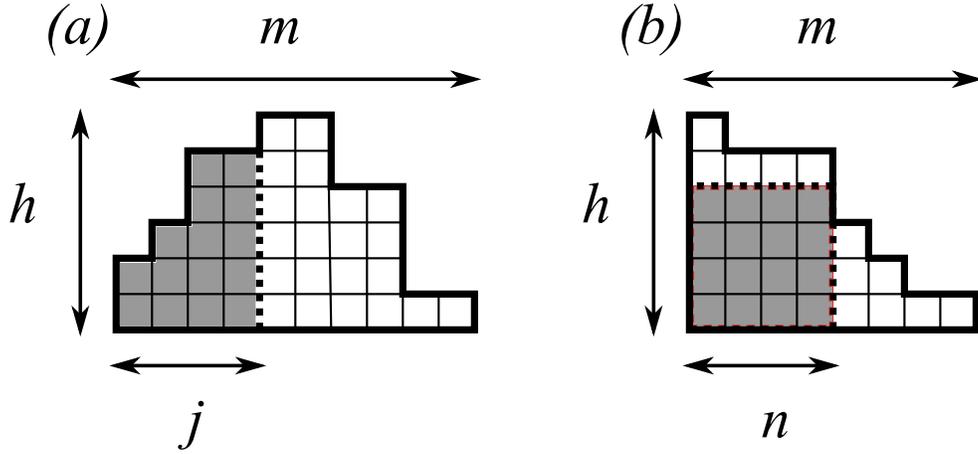


Figure 1: (a) An example of a stack polyomino of width  $m = 10$ , rise  $j = 4$ , height  $h = 6$  and area  $A = 37$ . Columns associated to the rise are shaded in grey. (b) An example of a Ferrers diagram of width  $m = 8$ , height  $h = 6$ , and area  $A = 28$ . The associated Durfee square is shaded in grey, with a red border.

and Eqn. (5) follows. Similarly, from Eqn. (8) it follows that

$$\begin{aligned} \Theta_0(x, q) &= (-x; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} (-x)^n}{(q; q)_n (-x; q)_n} \\ &= (-xq; q)_\infty \left[ 1 + x + \sum_{n=1}^{\infty} \frac{q^{n^2} (-x)^n}{(q; q)_n (-xq; q)_{n-1}} \right]. \end{aligned} \quad (11)$$

Hence  $\Theta_0(-\xi_0(q), q) = 0$  implies that

$$0 = 1 - \xi_0(q) + \sum_{n=1}^{\infty} \frac{q^{n^2} \xi_0(q)^n}{(q; q)_n (\xi_0(q)q; q)_{n-1}}, \quad (12)$$

and Eqn. (6) follows.  $\square$

The functional equations (5) and (6) are, broadly speaking, similar to those satisfied by different families of combinatorial trees. The interested reader should examine Chapter VII.4 of [5] and the more general theory of species [3] (due to Joyal [6]), for many examples of this.

A combinatorial interpretation of Eqn. (5) in Lemma 1 relates  $\xi_0(q)$  to stack polyominoes. A *stack polyomino* is defined by a unimodal sequence of positive integers  $h_1 \leq h_2 \leq h_3 \leq \dots \leq h_j < h_{j+1} \geq h_{j+2} \geq \dots \geq h_m$ . Given such a sequence, call  $m$  the width,  $j$  the rise ( $j$  may be zero),  $h \equiv h_{j+1}$  the height, and  $A = \sum_{i=1}^m h_i$  the area of the polyomino. We draw a stack polyomino as a set of columns of height  $h_1, \dots, h_m$  from left to right; an



Figure 2: Construction of stack polyominoes: a stack polyomino either has height one or can be obtained from a stack polyomino by appending a new row to the bottom of the stack polyomino.

example is given in Figure 1(a). Viewed in such a way, a unimodal sequence is equivalent to a stack polyomino of horizontal rows with no overhangs, hence the name.

Similarly, a combinatorial interpretation of Eqn. (6) in Lemma 1 will relate  $\xi_0(q)$  to Ferrers diagrams. A *Ferrers diagram* is defined by a decreasing sequence of positive integers  $m_1 \geq m_2 \geq m_3 \geq \dots \geq m_h$ . Given such a sequence, call  $h$  the height,  $m = m_1$  the width, and  $A = \sum_{i=1}^h m_i$  the area of the Ferrers diagram. We choose to draw Ferrers diagrams as a set of rows of lengths  $m_1, \dots, m_h$  from bottom to top; an example is shown in Figure 1(b). (The example shown in Figure 1(b) also has  $m_n = n$  for some integer  $n$ , a property that will become relevant later.)

The next lemma is an extension of results given in [7]. For general techniques for enumerating column-convex polyominoes, see also [4].

**Lemma 2.** *The generating function  $G(x, y, a, q)$  of stack polyominoes enumerated with respect to width ( $x$ ), height ( $y$ ), rise ( $a$ ), and total area ( $q$ ), is given by*

$$G(x, y, a, q) = \sum_{n=1}^{\infty} \frac{x(yq)^n}{(xq; q)_n (axq; q)_{n-1}}. \quad (13)$$

*Proof.* Any stack polyomino of height  $h > 1$  is uniquely constructed by adding a row to the bottom of a stack polyomino of height  $h - 1$ , as indicated in Figure 2.

This construction leads to a functional equation for the generating function  $G(x, y, a, q)$ . Stack polyominoes of height one are counted by

$$xyq + x^2yq^2 + x^3yq^3 + \dots = \frac{xyq}{1 - xq}. \quad (14)$$

Adding a row to the bottom of a stack polyomino of height  $h - 1$  is equivalent to increasing each column height of that polyomino by one and adding rows of non-negative length to each side. Increasing each column height corresponds to a change of  $G(x, y, a, q)$  to

$$yG(xq, y, a, q), \quad (15)$$

Adding rows to the left and right corresponds to multiplication with

$$1 + axq + a^2x^2q^2 + \dots = \frac{1}{1 - axq} \quad (16)$$

and

$$1 + xq + x^2q^2 + \dots = \frac{1}{1 - xq}, \quad (17)$$

respectively. Taken together, one finds that the generating function  $G(x) = G(x, y, a, q)$  satisfies the functional equation

$$G(x) = \frac{xyq}{1 - xq} + \frac{y}{(1 - xq)(1 - axq)}G(xq). \quad (18)$$

Iteration of this functional equation leads to the desired result.  $\square$

Combining the functional equation (5) of Lemma 1 and the combinatorial statement of Lemma 2 provides a connection of the root of the partial theta function with the enumeration of stack polyominoes.

**Corollary 3.** *The leading root  $x_0(q) = -\xi_0(q)$  of the partial theta function  $\Theta_0(x, q)$  satisfies*

$$\xi_0(q) = F(\xi_0(q), q), \quad (19)$$

where  $F(a, q) = 1 + G(1, 1, a, q)$  is the generating function of stack polyominoes augmented by the ‘empty polyomino’ of weight one, enumerated with respect to the rise ( $a$ ) and total area ( $q$ ).

*Proof.* Using Lemma 1 and Lemma 2 one finds immediately

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q; q)_n (\xi_0(q)q; q)_{n-1}} = 1 + G(1, 1, \xi_0(q), q). \quad (20)$$

$\square$

For an interpretation of this result, I now turn to the theory of species [3], and in particular to the combinatorics of Lagrange inversion in the context of combinatorial functional equations [3, Chapter 3]. The generating functions aspect of this theory is very well explained in [5, Section VII.4].

Given a combinatorial species of structures  $R$ , an  $R$ -enriched rooted tree on a finite set  $U$  is the data of (i) an arbitrary rooted tree on  $U$ , and (ii) an  $R$ -structure on the fiber of each vertex  $u \in U$ , in this rooted tree. Here, the *fiber* of a vertex  $u$  refers to the (possibly empty) set of immediate successors of  $u$ , when all edges of the rooted tree are oriented away from the root. The out-degree of  $u$  is the cardinality of its fiber.

In other words, an  $R$ -enriched rooted tree is an ordered rooted tree for which each vertex with out-degree  $d$  is decorated by an  $R$ -structure on a set of size  $d$ .

I shall make use of the following result.

**Lemma 4** (Theorem 2, Section 3.1 of [3]). *Let  $R$  be a species of structures. Then the species  $\mathcal{A}_R$  of  $R$ -enriched rooted trees is uniquely determined, up to isomorphism, by the combinatorial equation*

$$\mathcal{A}_R = X \cdot R(\mathcal{A}_R). \quad (21)$$

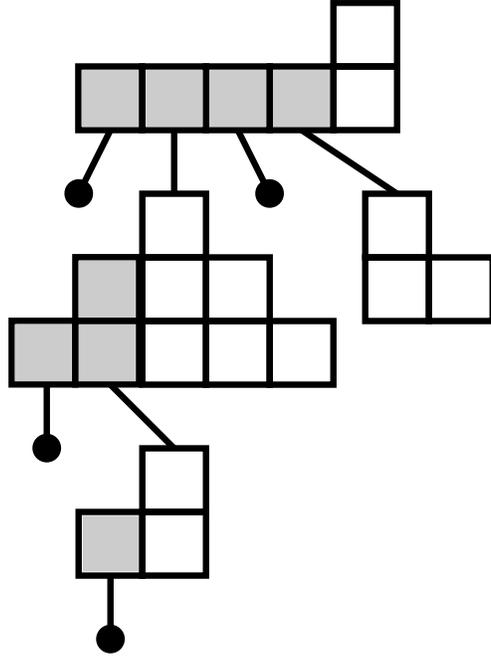


Figure 3: An example of an  $S_q$ -enriched rooted tree with 8 vertices and total area 21. Note that each column of the rise of the stack polyomino (shown in grey) is associated with a successor vertex in the tree. A leaf of the tree can be decorated by a stack polyomino of rise zero (a Ferrers diagram), or by an empty stack.

I now consider the species  $S_q$  of stack polyominoes augmented by the ‘empty polyomino’, weighted according to their area by the generating variable  $q$ , and with size given by the rise of the stack polyomino.  $S_q$ -enriched rooted trees are ordered rooted trees such that each vertex of the tree with out-degree  $d$  is decorated by a stack polyomino with rise  $d$ , and the weight of the tree is given by  $q^A$  where  $A$  is the sum of the areas of all of these polyominoes. Figure 3 shows an example of an  $S_q$ -enriched rooted tree.

I am now able to state the first theorem of this paper, which gives an explicit combinatorial interpretation of the coefficients of  $\xi_0(q)$ . Note that this implies immediately that these coefficients are positive integers.

**Theorem 5.** *Let  $S_q$  be the species of stack polyominoes augmented by the ‘empty polyomino’, weighted by area ( $q$ ), with size given by the rise. Then  $\xi_0(q)$  enumerates  $S_q$ -enriched rooted trees, weighted with respect to the total area of the stack polyominoes at the vertices of the tree.*

*Proof.* The species  $\mathcal{A}_{S_q}$  of  $S_q$ -enriched rooted trees satisfies  $\mathcal{A}_{S_q} = X \cdot S_q(\mathcal{A}_{S_q})$ . Correspondingly, its generating function  $A(t, q)$  satisfies

$$A(t, q) = tF(A(t, q), q), \quad (22)$$

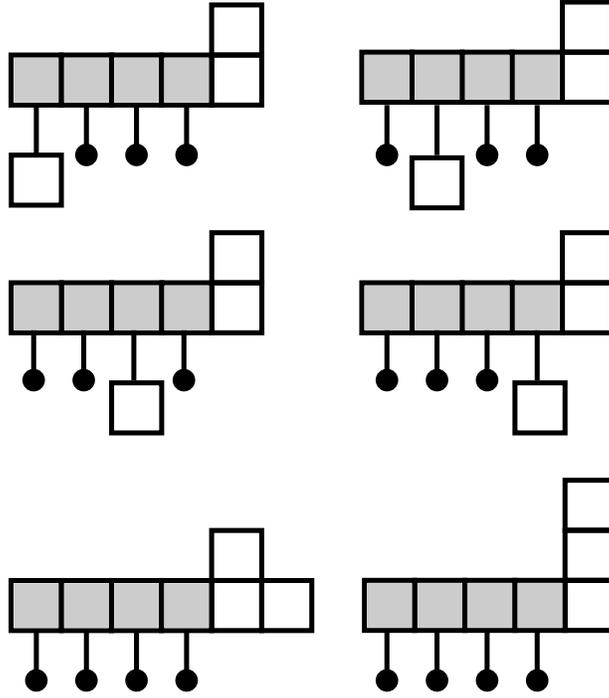


Figure 4: All six  $S_q$ -enriched rooted trees with 5 vertices and total area 7.

where  $t$  is the generating variable for the number of vertices in the tree. For  $t = 1$ , this equation reduces to  $A(1, q) = F(A(1, q), q)$ , whence one identifies  $\xi_0(q) = A(1, q)$  by Corollary 3.  $\square$

The generating function  $A(t, q)$  is a refinement of  $\xi_0(q)$ , enumerating enriched trees with respect to area and number of vertices, which could be studied in its own right. One computes easily

$$A(t, q) = t + tq + 2tq^2 + (t^2 + 3t)q^3 + (t^3 + 3t^2 + 5t)q^4 + (t^4 + 4t^3 + 9t^2 + 7t)q^5 + (t^5 + 5t^4 + 15t^3 + 20t^2 + 11t)q^6 + (t^6 + 6t^5 + 23t^4 + 44t^3 + 44t^2 + 15t)q^7 + \dots \quad (23)$$

For example, the occurrence of the monomial  $6t^5q^7$  in  $A(t, q)$  indicates there are six  $S_q$ -enriched rooted trees with 5 vertices and total area 7. These are shown in Figure 4.

As an immediate consequence of Theorem 5, one obtains a monotonicity result for the coefficients of  $\xi_0(q)$ .

**Corollary 6.** *The coefficients of the power series  $\xi_0(q)$  are monotonically increasing.*

*Proof.* Monotonicity follows from the existence of an injection of  $S_q$ -enriched rooted trees with total area  $A$  to  $S_q$ -enriched rooted trees with total area  $A + 1$ , which is defined as follows.

The only  $S_q$ -enriched rooted tree with zero total area is a tree with a single vertex, enriched by the empty stack polyomino. Map this tree to an  $S_q$ -enriched rooted tree with

total area one by enriching its vertex by a square and appending a leaf enriched by an empty stack.

Now consider an  $S_q$ -enriched rooted tree with non-zero total area  $A$ . Then the root of this tree is enriched by a stack polyomino of non-zero area. Map this tree to an  $S_q$ -enriched rooted tree with total area  $A + 1$  by appending a square to the right of the bottom row of the stack polyomino at its root. This increases its area by one, and as this operation does not affect the rise of this stack polyomino, the resulting tree is again an  $S_q$ -enriched rooted tree.

Clearly, different  $S_q$ -enriched trees with total area  $A$  get mapped to different  $S_q$ -enriched trees with total area  $A + 1$ , hence this mapping is injective.  $\square$

A combinatorial interpretation of Eqn. (6) in Lemma 1 provides another description of  $\xi_0(q)$ , relating it to partitions of integers with the particular property that there exists an integer  $n$  such that the  $n$ -th part of the partition has exactly size  $n$ .

We remind that the generating function for Ferrers diagrams enumerated with respect to width ( $x$ ), height ( $y$ ), and area ( $q$ ) is given by

$$H(x, y, q) = \sum_{n=1}^{\infty} \frac{x^n}{(yq; q)_n} = \sum_{n=1}^{\infty} \frac{(xy)^n q^{n^2}}{(xq; q)_n (yq; q)_n} . \quad (24)$$

Here, the first expression is obtained by summing over all columns of width  $n$ , and the second expression is obtained by constructing Ferrers diagrams by adding Ferrers diagrams to the top and right of a square of size  $n$ , which is also known as the Durfee square [9] associated to that Ferrers diagram. A close inspection of Eqn. (6) indicates that a modification of the second sum in Eqn. (24) is needed, using a construction where the single square immediately to the right of the top row of the Durfee square is absent from the Ferrers diagram. This condition is equivalent to saying that the  $n$ -th largest row of the Ferrers diagram has length  $n$  for some positive integer  $n$ .

**Lemma 7.** *The generating function  $\tilde{G}(x, y, q)$  of Ferrers diagrams with  $n$ -th largest row having length  $n$  for some positive integer  $n$ , enumerated with respect to width ( $x$ ), height ( $y$ ), and total area ( $q$ ), is given by*

$$\tilde{G}(x, y, q) = \sum_{n=1}^{\infty} \frac{(xy)^n q^{n^2}}{(yq; q)_n (xq; q)_{n-1}} . \quad (25)$$

*Proof.* Partitioning Ferrers diagrams by their associated Durfee square, the side length of which is given by

$$n = \max\{i : m_i \geq i\} , \quad (26)$$

one obtains Ferrers diagrams for which the  $n$ -th largest row has precisely length  $n$  by adding Ferrers diagrams of width  $\leq n$  to the top of the square and adding Ferrers diagrams of height  $\leq n - 1$  to the right of the square.

The weight of a Durfee square of fixed side length  $n$  is  $(xy)^n q^{n^2}$ . Adding Ferrers diagrams of height  $\leq n-1$  to the right hand side of the square corresponds to multiplication with

$$\frac{1}{1-xq^{n-1}} \cdot \frac{1}{1-xq^{n-2}} \cdots \frac{1}{1-xq^2} \cdot \frac{1}{1-xq} = \frac{1}{(xq; q)_{n-1}}. \quad (27)$$

Here, a factor  $1/(1-xq^k)$  corresponds to the addition of an arbitrary number of columns of height  $k$ , and clearly  $k$  can range from 1 to  $n-1$ . Similarly, adding Ferrers diagrams of width  $\leq n$  to the top of the square corresponds to multiplication with

$$\frac{1}{1-yq^n} \cdot \frac{1}{1-yq^{n-1}} \cdots \frac{1}{1-yq^2} \cdot \frac{1}{1-yq} = \frac{1}{(yq; q)_n}. \quad (28)$$

Here, a factor  $1/(1-yq^k)$  corresponds to the addition of an arbitrary number of rows of width  $k$ , and clearly  $k$  can range from 1 to  $n$ . A Durfee square of fixed side length  $n$  gives therefore rise to an associated generating function

$$\tilde{G}_n(x, y, q) = \frac{(xy)^n q^{n^2}}{(yq; q)_n (xq; q)_{n-1}}, \quad (29)$$

and summing over all side lengths  $n \geq 1$  gives

$$\tilde{G}(x, y, q) = \sum_{n=1}^{\infty} \tilde{G}_n(x, y, q) \quad (30)$$

as desired. □

Combining the functional equation (6) of Lemma 1 and the combinatorial statement of Lemma 6 provides a connection of the root of the partial theta function with the enumeration of Ferrers diagrams.

To achieve this, I consider the species  $F_q$  of Ferrers diagrams with  $n$ -th largest row having length  $n$  for some integer  $n$ , augmented by the ‘empty polyomino’, weighted according to their area by the generating variable  $q$ , and with size given by the width of the polyomino.  $F_q$ -enriched rooted trees are ordered rooted trees such that each vertex of the tree with out-degree  $d$  is decorated by a Ferrers diagram of width  $d$ , and the weight of the tree is given by  $q^A$  where  $A$  is the sum of the areas of all of these polyominoes. Figure 5 shows an example of an  $F_q$ -enriched rooted tree.

**Corollary 8.** *The leading root  $x_0(q) = -\xi_0(q)$  of the partial theta function  $\Theta_0(x, q)$  satisfies*

$$\xi_0(q) = \tilde{F}(\xi_0(q), q), \quad (31)$$

where  $\tilde{F}(a, q) = 1 + \tilde{G}(a, 1, q)$  is the generating function of Ferrers diagrams with  $n$ -th largest row having length  $n$  for some integer  $n$ , augmented by the ‘empty polyomino’ and enumerated with respect to height ( $a$ ) and total area ( $q$ ).

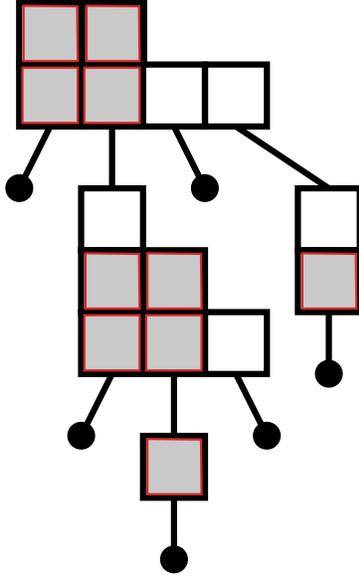


Figure 5: An example of an  $F_q$ -enriched rooted tree with 10 vertices and total area 15. Note that the out-degree of each vertex is equal to the width of the Ferrers diagram associated with that vertex. The occurring Ferrers diagrams are such that their  $n$ -th largest row has length  $n$  for some integer  $n$  (the associated Durfee square is indicated in grey, with a red border). In particular, all leaves are decorated by empty Ferrers diagrams.

*Proof.* Combining Lemma 1 and Lemma 6 one finds

$$\xi_0(q) = 1 + \sum_{n=1}^{\infty} \frac{\xi_0(q)^n q^{n^2}}{(q; q)_n (\xi_0(q)q; q)_{n-1}} = 1 + \tilde{G}(\xi_0(q), 1, q). \quad (32)$$

□

This leads to the second theorem of this paper, which gives a different explicit combinatorial interpretation of the coefficients of  $\xi_0(q)$ . (Of course this theorem also implies that these coefficients are positive integers.)

**Theorem 9.** *Let  $F_q$  be the species of Ferrers diagrams with  $n$ -th largest row having length  $n$  for some integer  $n$ , weighted by area ( $q$ ), with size given by the width of the Ferrers diagram, augmented by the ‘empty polyomino’. Then  $\xi_0(q)$  enumerates  $F_q$ -enriched rooted trees with respect to the total area of the Ferrers diagrams at the vertices of the tree.*

*Proof.* The species  $\mathcal{A}_{F_q}$  of  $F_q$ -enriched rooted trees satisfies  $\mathcal{A}_{F_q} = X \cdot F_q(\mathcal{A}_{F_q})$ . Correspondingly, its generating function  $\tilde{A}(x, q)$  satisfies

$$\tilde{A}(t, q) = t\tilde{F}(\tilde{A}(t, q), q), \quad (33)$$

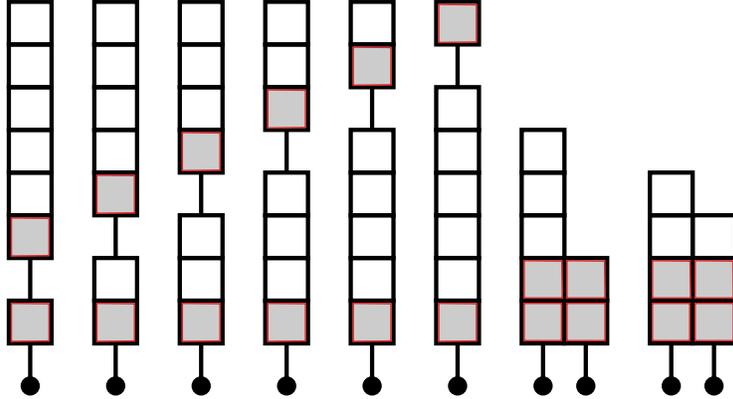


Figure 6: All eight  $F_q$ -enriched rooted trees with 3 vertices and total area 7.

where  $x$  is the generating variable for the size of the trees. For  $t = 1$ , this equation reduces to  $\tilde{A}(1, q) = 1 + \tilde{F}(\tilde{A}(1, q), q)$ , whence one identifies  $\xi_0(q) = \tilde{A}(1, q)$ .  $\square$

Figure 5 shows an example of an  $F_q$ -enriched rooted tree. The generating function  $\tilde{A}(t, q)$  is a refinement of  $\xi_0(q)$  clearly different from  $A(t, q)$ . One computes easily

$$\begin{aligned} \tilde{A}(t, q) = & t + t^2q + (t^3 + t^2)q^2 + (t^4 + 2t^3 + t^2)q^3 + (t^5 + 3t^4 + 4t^3 + t^2)q^4 + (t^6 + 4t^5 + 10t^4 + 5t^3 + t^2)q^5 \\ & + (t^7 + 5t^6 + 21t^5 + 17t^4 + 7t^3 + t^2)q^6 + (t^8 + 6t^7 + 41t^6 + 47t^5 + 29t^4 + 8t^3 + t^2)q^7 + \dots \end{aligned} \quad (34)$$

For example, the occurrence of the monomial  $8t^3q^7$  in  $\tilde{A}(t, q)$  indicates there are eight  $F_q$ -enriched rooted trees with 3 vertices and total area 7. These are shown in Figure 6.

An immediate consequence of Theorems 5 and 9 is the following corollary.

**Corollary 10.**  *$S_q$ -enriched rooted trees with fixed total area  $A$  and  $F_q$ -enriched rooted trees with fixed total area  $A$  are equinumerous.*

However, this is just a special case of a much more general result. Given that the tree structures in Lemma 2 and Lemma 4 are related to the iteration of the two functional equations (5) and (6) for  $\xi_0(q)$  from Lemma 1, respectively, one can generalise Theorems 5 and 9 immediately to an infinite family of equinumerous sets of trees.

To see this, note that  $\xi_0(q)$  satisfies not only

$$\xi_0(q) = F(\xi_0(q), q), \text{ and} \quad (35)$$

$$\xi_0(q) = \tilde{F}(\xi_0(q), q), \quad (36)$$

but also

$$\xi_0(q) = F(F(\xi_0(q), q), q), \quad (37)$$

$$\xi_0(q) = F(\tilde{F}(\xi_0(q), q), q), \quad (38)$$

$$\xi_0(q) = \tilde{F}(F(\xi_0(q), q), q), \quad (39)$$

$$\xi_0(q) = \tilde{F}(\tilde{F}(\xi_0(q), q), q), \quad (40)$$

and so forth, and iteration of each of these leads to a different combinatorial model. More generally, I obtain Theorem 11.

**Theorem 11.** *Let  $F_q^{(0)}(a) = 1 + G(1, 1, a, q)$  be the generating function of stack polyominoes enumerated with respect to rise ( $a$ ) and total area ( $q$ ), and let  $F_q^{(1)}(a) = 1 + \tilde{G}(a, 1, q)$  be the generating function of Ferrers diagrams with  $n$ -th largest column having length  $n$  for some integer  $n$ , enumerated with respect to width ( $a$ ) and total area ( $q$ ).*

*Let  $\sigma = \{\sigma_0, \dots, \sigma_N\} \in \{0, 1\}^{N+1}$  for  $N \geq 0$ . Then*

$$\xi_0^\sigma(q) = F_q^{(\sigma_0)} \circ F_q^{(\sigma_1)} \circ \dots \circ F_q^{(\sigma_N)}(0) \quad (41)$$

*enumerates rooted trees of height at most  $N$ , enriched by  $S_q$  at level  $i$  if  $\sigma_i = 0$  and enriched by  $F_q$  at level  $i$  if  $\sigma_i = 1$ , weighted with respect to area (level 0 is the root).*

*Moreover, given  $\sigma \in \{0, 1\}^{\mathbb{N}}$ ,  $\xi_0(q)$  enumerates rooted trees enriched by  $S_q$  at level  $i$  if  $\sigma_i = 0$  and enriched by  $F_q$  at level  $i$  if  $\sigma_i = 1$ , weighted with respect to area. In particular, sets of trees enriched with respect to any  $\sigma \in \{0, 1\}^{\mathbb{N}}$  are equinumerous for fixed total area  $A$ .*

*Proof.* Let  $R_q^{(0)} = S_q$  be the species of stack polyominoes augmented by the ‘empty polyomino’, weighted by area ( $q$ ), with size give by the rise, and let  $R_q^{(1)} = F_q$  be the species of Ferrers diagrams with  $n$ -th largest row having length  $n$  for some integer  $n$ , augmented by the ‘empty polyomino’, weighted by area ( $q$ ), with size give by the width. Let  $\sigma \in \{0, 1\}^{N+1}$ . Then the combinatorial expression

$$X \cdot R_q^{(\sigma_0)}(X \cdot R_q^{(\sigma_1)}(\dots X \cdot R_q^{(\sigma_N)}(0) \dots)) \quad (42)$$

encodes trees of height at most  $N$ , enriched by  $R_q^{(\sigma_i)}$  at level  $i$ . Hence

$$A_\sigma(t, q) = tF_q^{(\sigma_0)}(tF_q^{(\sigma_1)}(\dots tF_q^{(\sigma_N)}(0) \dots)) \quad (43)$$

enumerates these trees with respect to number of vertices ( $t$ ) and total area ( $q$ ). Letting  $t = 1$ , Eqn. (41) follows.

Next, let  $\sigma \in \{0, 1\}^{\mathbb{N}}$ . For any finite  $N$ , the restrictions of

$$X \cdot R_q^{(\sigma_0)}(X \cdot R_q^{(\sigma_1)}(\dots X \cdot R_q^{(\sigma_M)}(0) \dots)), \quad M \geq N \quad (44)$$

to trees of height at most  $N$  coincide. Hence, letting  $N \rightarrow \infty$ , the claim follows.  $\square$

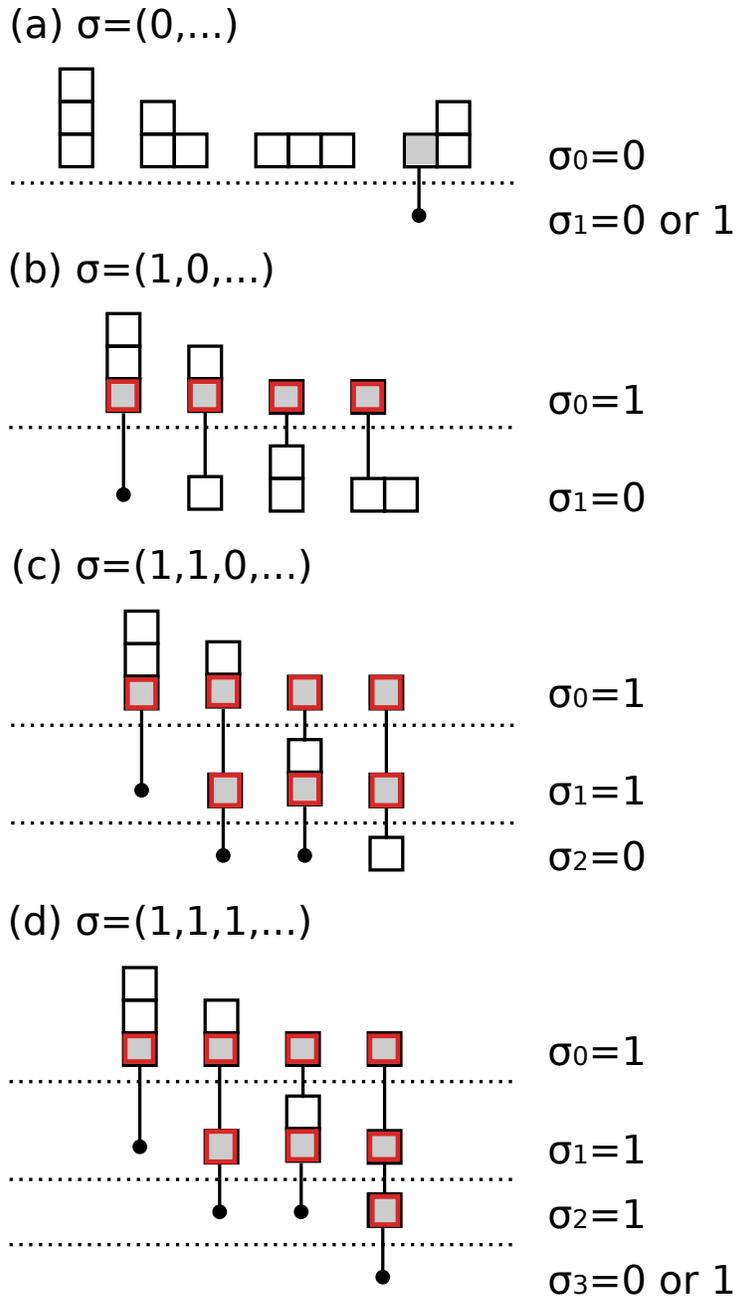


Figure 7: Enriched rooted trees with total area 3 for different choices of  $\sigma$ : (a) trees enriched with stack polyominoes at level 0, (b) trees enriched with Ferrers diagrams at level 0 and stack polyominoes at level 1, (c) trees enriched with Ferrers diagrams at levels 0 and 1, and stack polyominoes at level 2, (d) trees enriched with Ferrers diagrams at levels 0, 1, and 2.

Figure 7 shows enriched rooted trees with total area 3 for different choices of  $\sigma \in \{0, 1\}^{\mathbb{N}}$ . For each choice of  $\sigma$ , there are four enriched trees with total area 3. Note that the number of vertices in the trees differ, corresponding to different expressions for the coefficient polynomials of  $q^3$  in  $A_\sigma(t, q)$ :

(a)  $\sigma = (0, \dots)$  implies

$$A_\sigma(t, q) = t + tq + 2tq^2 + (t^2 + 3t)q^3 + \dots, \quad (45)$$

(b)  $\sigma = (1, 0, \dots)$  implies

$$A_\sigma(t, q) = t + t^2q + 2t^2q^2 + 4t^2q^3 + \dots, \quad (46)$$

(c)  $\sigma = (1, 1, 0, \dots)$  implies

$$A_\sigma(t, q) = t + t^2q + (t^3 + t^2)q^2 + (3t^3 + t^2)q^3 + \dots, \quad (47)$$

and

(d)  $\sigma = (1, 1, 1, \dots)$  implies

$$A_\sigma(t, q) = t + t^2q + (t^3 + t^2)q^2 + (t^4 + 2t^3 + t^2)q^3 + \dots. \quad (48)$$

It would be highly interesting to find direct bijections between trees generated by any two different choices of  $\sigma \in \{0, 1\}^{\mathbb{N}}$ .

Note that Theorem 11 provides a combinatorial interpretation of the iterations in remarks 4 and 5 after Proposition 3.1 in [8], as these are simply  $\xi_0^{(n)}(q) = (F_q^{(0)})^n(0)$  and  $\xi_0^{(n)}(q) = (F_q^{(1)})^n(1) = (F_q^{(1)})^{n+1}(0)$ , respectively.

The iteration used in Proposition 3.1 in [8] is given by

$$\xi_0^{(n)}(q) = (F_q^{(1)})^n(1). \quad (49)$$

Starting the iteration with  $\xi_0^{(0)}(q) = 1$  instead of 0 implies that  $\xi_0^{(n)}(q) = (F_q^{(1)})^n(1)$  counts rooted trees of height at most  $n - 1$ , enriched with stack polyominoes, with the constraint relaxed by allowing enrichment with *arbitrary* stack polyominoes at level  $n - 1$ . Clearly this also immediately generalises to general choices of  $\sigma$ .

Note that it was natural in the proofs to introduce the generating variable  $t$  corresponding to the number of vertices in the trees. It is tempting to ask what the meaning of this variable (or, indeed, any other generating variable related to the tree structure, e.g. counting the number of leaves in the tree) would be in a generalised theta function which has  $-A_\sigma(t, q)$  as its leading root.

I shall close with a remark on the singularity structure of  $\xi_0(x)$ . From [5, Section VII.4] it follows that one expects the presence of a generic square root singularity for  $\xi_0(q)$ . Indeed

a simple, albeit non-rigorous numerical analysis indicates that the  $n$ -th coefficient of  $\xi_0(q)$  grows asymptotically as

$$[q^n]\xi_0(q) \sim A\mu^n n^{-3/2} \quad \text{as } n \rightarrow \infty, \quad (50)$$

where  $\mu = 3.2336366652450763163646925293871348350211819091413196994020357434 \dots$ , consistent with such a square singularity.

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