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Exact solution of the discrete (1+1)-dimensional RSOS model in a slit with field and wall interactions

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Abstract
We present the solution of a linear restricted solid-on-solid (RSOS) model confined to a slit. We include a field-like energy, which equivalently weights the area under the interface, and also include independent interaction terms with both walls. This model can also be mapped to a lattice polymer model of Motzkin paths in a slit interacting with both walls including an osmotic pressure. This work generalizes the previous work on the RSOS model in the half-plane which has a solution that was shown recently to exhibit a novel mathematical structure involving basic hypergeometric functions ${}_3\phi_2$. Because of the mathematical relationship between the half-plane and slit this work hence effectively explores the underlying q-orthogonal polynomial structure to that solution. It also generalizes two other recent works: one on Dyck paths weighted with an osmotic pressure in a slit and another concerning Motzkin paths without an osmotic pressure term in a slit.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Solid-on-solid (SOS) models describe the interface between low-temperature phases, originally in magnetic systems such as Ising-like models [1, 2], though now more generally. They are effectively directed models in $d + 1$ dimensions. The configurations involved in the linear (1+1)-dimensional case, modelling the interface in a two-dimensional system, have also been used to model the backbone of a polymer in the solution [3]. The critical phenomena associated with this model describe wetting transitions of the interface with a wall [2]. For the SOS model the phase diagram contains a wetting transition at finite temperature T_w for the zero field, and complete wetting occurs taking the limit $H \to 0$ for $T \geqslant T_w$ [4]. The term in the Hamiltonian that leads to the magnetic field in this model can be reinterpreted as an osmotic pressure when considering self-avoiding polygon models of vesicles [5]. The basic SOS model is naturally described in the half-plane but it is also natural to describe it in the confined geometry of the slit.

The linear SOS model with the magnetic field and wall interaction was solved in [4]. Recently the restricted SOS (RSOS), where the interface takes on a restricted subset of configurations, was solved with the same interactions of field and single wall interaction in the half-plane [8]. This has proven to be mathematically quite interesting as both the method of solution and the functions involved were novel. It was found that the solution could be expressed as ratios of linear combinations of terms involving the basic hypergeometric function ${}_3\phi_2$. Recently the polymer models of Dyck paths [6] and Motzkin paths [9] in a slit with separate interactions with both surfaces have also been considered, without field-like terms. Here the solutions in the slit prove interesting both mathematically and physically. They are of interest physically because the infinite slit limit was shown to be subtly different to the half-plane, realizing a separate phase diagram [6]. Mathematically the slit exposes the orthogonal polynomial structure of the problem and uncovers hidden combinatorial relationships [9]. Finally, Dyck paths in a slit with wall interactions and weighted by the area under the path, equivalent to a field term in the SOS models, have only recently been analysed [7], and show a rich q-orthogonal polynomial structure. To further explore this area of research here we consider the RSOS model in a slit geometry with both separate wall interactions and a field/osmotic pressure term in the energy. We derive the novel q-orthogonal polynomials for this problem which give us the exact solution of the generating function.

2. The RSOS model in a slit

The RSOS model we analyse can be described as follows. Consider a two-dimensional square lattice in a slit of width (or thickness) $w \geqslant 1$. For each column i of the surface a segment of the interface is placed on the horizontal link at height $0 \leqslant r_i \leqslant w$, and successive segments are joined by vertical segments to form a partially directed interface with no overhangs. The configurations are given by the energy

$$-\beta E = -K \sum_{i=1} |r_i - r_{i-1}| - H \sum_{i=1} r_i + B_0 \sum_{i=1} \delta_{r_i,0} + B_w \sum_{i=1} \delta_{r_i,w}. \qquad (2.1)$$

As in [8], we discuss the RSOS model in terms of lattice paths. An RSOS path is a partially directed self-avoiding path with no steps into the negative x-direction and no successive vertical steps. To be precise, an RSOS path of length N with heights r_0 to r_N has horizontal steps at heights $r_1, \ldots, r_N$, and vertical steps between heights r_{i-1} and r_i for $i = 1, \ldots, N$, but no horizontal step associated with r_0. This means that an RSOS path starts at height r_0 with either a horizontal step (if $r_1 = r_0$) or a vertical step (if $r_1 \neq r_0$), but must end at height r_N with a horizontal step. Figure 1 shows an example.

The partition function for the RSOS paths of length N in a slit of width $w \geqslant 1$ with ends fixed at heights $r_0 \geqslant 0$ and $r_N \geqslant 0$, respectively, is given by

$$Z_1^w(r_0; r_1) = \begin{cases} \exp(-\beta E(r_0; r_1)), & |r_0 - r_1| \leqslant 1 \\ 0, & |r_0 - r_1| > 1, \end{cases} \qquad (2.2)$$

and

$$Z_N^w(r_0; r_N) = \sum_{\substack{0 \leqslant r_1, \ldots, r_{N-1} \leqslant w \\ |r_i - r_{i-1}| \leqslant 1}} \exp(-\beta E(r_0; r_1, \ldots, r_N)), \qquad N = 2, 3, \ldots, \qquad (2.3)$$

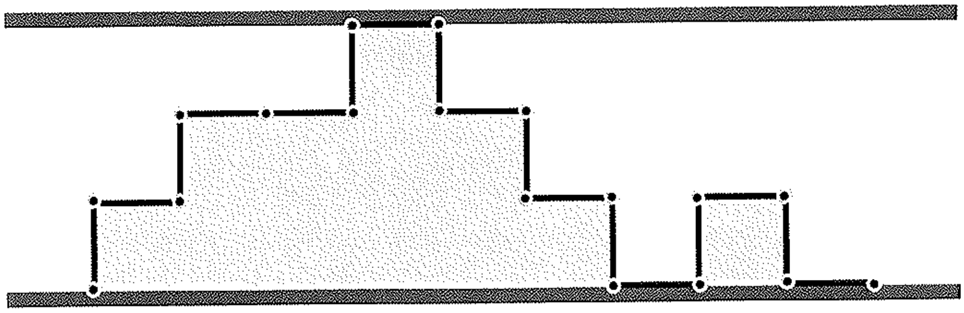

Figure 1. A typical RSOS configuration beginning on the surface and finishing on the surface with a horizontal step: each horizontal step has been assigned a weight x, each vertical step a weight y, each unit of area a weight q, each step that touches the bottom surface an additional weight a while each step that touches the top surface an additional weight b. The length of the configuration shown here is $N = 9$ in a slit of width $w = 3$, and the heights are $r_0 = 0, r_1 = 1, r_2 = r_3 = 2, r_4 = 3, r_5 = 2, r_6 = 1, r_7 = 0, r_8 = 1$ and $r_9 = 0$. The weight of this configuration equals $x^9 y^8 q^{12} a^2 b$.

where

$$-\beta E(r_0; r_1, \ldots, r_N) = -K\sum_{i=1}^{N}|r_i - r_{i-1}| - H\sum_{i=1}^{N} r_i + B_0\sum_{i=1}^{N}\delta_{r_i,0} + B_w\sum_{i=1}^{N}\delta_{r_i,w}. \qquad (2.4)$$

Here, we shall consider paths with both ends attached to the surface, i.e. we shall focus on the partition function

$$Z_N^w = Z_N^w(0; 0). \qquad (2.5)$$

We define

$$y = \exp(-K), \qquad q = \exp(-H), \qquad a = \exp(B_0) \qquad \text{and} \qquad b = \exp(B_w), \qquad (2.6)$$

so y is a temperature-like variable, q is a magnetic field-like variable, and a and b are binding energy-like variables, and we write

$$Z_N^w = Z_N^w(y, q, a, b). \qquad (2.7)$$

The (reduced) free energy is then

$$\kappa(w; y, q, a, b) = \lim_{N\to\infty}\frac{1}{N}\log Z_N^w(y, q, a, b). \qquad (2.8)$$

The argument showing the existence of this limit can be summarized as follows. Given a strip of width w, concatenating two RSOS configurations of length N_1 and N_2 respectively, by identifying the final vertex of the first configuration with the first vertex of the second configuration, one obtains an RSOS configuration of length N_1+N_2. Under this concatenation, the number of contacts with either wall is simply additive, as is the number of vertical steps and total area. This immediately implies supermultiplicativity of the partition function, i.e.

$$Z_{N_1+N_2}^w(y, q, a, b) \geqslant Z_{N_1}^w(y, q, a, b) Z_{N_2}^w(y, q, a, b), \qquad (2.9)$$

whenever y, q, a and b are non-negative. This inequality, together with the upper bound

$$Z_N^w(y, q, a, b) \leqslant ((1 + 2y)(1 + q^w)(1 + a)(1 + b))^N \qquad (2.10)$$

implies the existence of $\lim_{N\to\infty}\frac{1}{N}\log Z_N^w(y, q, a, b)$ by a standard subadditivity argument. We subsequently calculate the generating function explicitly below, which then also implies the existence of this limit.

Define a generalized (grand canonical) partition function, or, simply, generating function, as

$$G_w(x, y, q, a, b) = 1 + \sum_{N=1}^{\infty} x^N Z_N^w(y, q, a, b). \tag{2.11}$$

Thus, the radius of convergence $x_c(w; y, q, a, b)$ of $G^w(x, y, q, a, b)$ with respect to the series expansion in x can be identified as $\exp(-\kappa(w; y, q, a, b))$; hence,

$$\kappa(w; y, q, a, b) = -\log x_c(w; y, q, a, b). \tag{2.12}$$

It is convenient to consider G as a combinatorial generating function for RSOS paths, where x, y, q, a and b are the counting variables for appropriate properties of those paths. Interpreted in such a way, x and y are the weights of horizontal and vertical steps, respectively, q is the weight for each unit of area enclosed by the RSOS path and the x-axis, a is an additional weight for each step that touches the bottom surface while b is an additional weight for each step that touches the top surface. For example, the weight of the configuration in figure 1 is $x^9 y^8 q^{12} a^2 b$.

If we send $w \to \infty$ then we recover the generating function of the half-plane:

$$G^{\mathrm{hp}}(x, y, q, a) = 1 + \sum_{N=1}^{\infty} x^N Z_N^{\mathrm{hp}}(y, q, a), \tag{2.13}$$

where $Z_N^{\mathrm{hp}}(y, q, a) = Z_N^w(y, q, a, b)$ for $w > N$, noting that the paths can have no more vertical steps than horizontal steps in an RSOS path, and thus cannot visit the top surface when $w > N$. Therefore, when $w > N$ the partition function $Z_N^w(y, q, a, b)$ becomes independent of b, and thus independent of B_w.

We find easily the first few terms of G^{hp} as a series expansion in x:

$$G^{\mathrm{hp}}(x, y, q, a) = 1 + ax + (a^2 + ay^2 q)x^2 + \cdots, \tag{2.14}$$

where the constant term corresponds to a zero-step path starting and ending at height zero with weight one.

Note that the radius of convergence of the half-plane generating function $x_c^{\mathrm{hp}}(y, q, a, b)$ is not *a priori* the limit of the slit $x_c(w; y, q, a, b)$ as was demonstrated in [6] for the corresponding Dyck path problem with $q = 1$.

3. Exact solution for the generating function

The key to the solution is a combinatorial decomposition of RSOS paths which leads to a functional equation for the generating function G_w. This decomposition is done with respect to the left-most horizontal step touching the surface at height zero, and is shown diagrammatically in figure 2.

We distinguish three cases.

(a) The RSOS path has zero length, and there is no horizontal step at height zero. The contribution to the generating function is 1.

(b) The RSOS path starts with a horizontal step, which therefore is at height zero. The rest of this path is again an RSOS path. The contribution to the generating function is $\kappa x G_w(x, y, q, \kappa)$.

(c) The RSOS path starts with a vertical step. Then there will be a left-most horizontal step at height zero, and removing this step cuts the path into two pieces. The left path starts with a vertical and horizontal step, followed by an RSOS path starting and

$$G_w(x,a) = 1 + axG_w(x,a) + ax^2y^2qG_{w-1}(qx,1)G_w(x,a)$$

Figure 2. The diagrammatic form of the functional equations for RSOS paths, indicating the combinatorial decomposition of RSOS paths.

ending at height one and not touching the surface, followed by a vertical step to height zero. This left path is effectively in a slit of width $w-1$. The right path is again an RSOS path in the slit of width w. The contribution to the generating function is $yxqG_{w-1}(qx,y,q,1)y\kappa xG_w(x,y,q,\kappa)$.

Putting together, this decomposition leads to a functional-recurrence equation for the generating function

$$G_w(x,y,q,a,b) = 1 + axG_w(x,y,q,a,b) + aqx^2y^2G_{w-1}(qx,y,q,1,b)G_w(x,y,q,a,b), \tag{3.1}$$

with an 'initial condition'

$$G_0(x,y,q,a,b) = \frac{b}{1-abx}. \tag{3.2}$$

If we send $w \to \infty$ then we recover the functional equation of the half-plane

$$G^{\mathrm{hp}}(x,y,q,a) = 1 + axG^{\mathrm{hp}}(x,y,q,a) + aqx^2y^2G^{\mathrm{hp}}(qx,y,q,1)G^{\mathrm{hp}}(x,y,q,a), \tag{3.3}$$

whose solution is given in [8].

We can rewrite (3.1) as

$$G_w(x,y,q,a,b) = \frac{1}{1-ax-aqx^2y^2G_{w-1}(qx,y,q,1,b)}. \tag{3.4}$$

For general values of q, a formal iteration of (3.4) leads to a continued fraction expansion

$$G_w(x,y,q,a,b) = \cfrac{1}{1-ax-\cfrac{aqx^2y^2}{1-qx-\cfrac{q^3x^2y^2}{1-q^2x-\cfrac{q^5x^2y^2}{1-q^3x-\cfrac{q^7x^2y^2}{\ddots\,\cfrac{}{1-q^{w-1}x-\cfrac{bq^{2w-1}x^2y^2}{1-bq^wx}}}}}}}. \tag{3.5}$$

However, there is a non-trivial method to solve the functional equation for G as a ratio of q-orthogonal polynomials. It is clear that the generating function can be written as a rational function

$$G_w(x,y,q,a,b) = \frac{P_w(x,y,q,a,b)}{Q_w(x,y,q,a,b)} \tag{3.6}$$

though it does not simply follow to write expressions for these. It does however follow from the theories of continued fractions and orthogonal polynomials (see pages 256–57 of

Andrews *et al* [10]) that both the numerator P_w and denominator Q_w of the generating function satisfy recursions

$$P_w(x,y,q,a,b) = \begin{cases} b, & w=0 \\ 1-bqx, & w=1 \\ (1-bq^w x)P_{w-1}(x,y,q,a,1) - bq^{2w-1}x^2y^2P_{w-2}(x,y,q,a,1) & w\geqslant 2 \end{cases} \tag{3.7}$$

and

$$Q_w(x,y,q,a,b) = \begin{cases} 1-abx, & w=0 \\ 1-bqx-ax(1-bqx(1-y^2)), & w=1 \\ -bq^w x Q_{w-1}(x,y,q,a,1) - bq^{2w-1}x^2y^2Q_{w-2}(x,y,q,a,1) & w\geqslant 2. \end{cases} \tag{3.8}$$

These are the functions $Q_w(x,y,q,a,b)$ which are the q-orthogonal polynomials referred to in the introduction.

One can immediately note that

$$P_w(x,y,q,a,b) = Q_w(x,y,q,0,b) \tag{3.9}$$

so that

$$G_w(x,y,q,a,b) = \frac{Q_w(x,y,q,0,b)}{Q_w(x,y,q,a,b)}. \tag{3.10}$$

We now form the width generating function for the denominator as

$$W(t,x,y,q,a,b) = \sum_{w=0}^{\infty} Q_w(x,y,q,a,b)t^w \tag{3.11}$$

and find a functional equation for $W(t)$ from recurrence (3.8) as

$$\begin{aligned} W(t,x,y,q,a,b) &= 1-abx-abqx^2y^2t+tW(t,x,y,q,a,1) \\ &\quad -bqxtW(qt,x,y,q,a,1) - bq^3x^2y^2t^2W(q^2t,x,y,q,a,1). \end{aligned} \tag{3.12}$$

Unlike in the case of Dyck paths in a slit [7], one cannot solve for $W(t) \equiv W(t,x,y,q,a,1)$ by direct iteration, as the functional equation involves $W(t)$, $W(qt)$ and $W(q^2t)$. So let us return to $Q_w(x,y,q,a,b)$ and consider the case $b=1$, as this is all that is required to find the full solution. Let

$$R_w \equiv Q_w(x,y,q,a,1). \tag{3.13}$$

From recurrence (3.8) we have for $w \geqslant 2$ that

$$R_w = (1-q^w x)R_{w-1} - q^{2w-1}x^2y^2R_{w-2} \tag{3.14}$$

with initial conditions

$$\begin{aligned} R_0 &= 1-ax, \\ R_1 &= 1-qx-ax(1-bqx(1-y^2)). \end{aligned} \tag{3.15}$$

Attempting to mimic the aspects of the half-plane solution [8], we define S_w via

$$R_w = (-1)^w q^{w(w+1)/2}x^w S_w. \tag{3.16}$$

This gives us the recurrence

$$q^w x(S_w - S_{w-1} + y^2S_{w-2}) + S_{w-1} = 0 \tag{3.17}$$

with initial conditions

$$S_0 = 1 - ax, \\ -qxS_1 = 1 - qx - ax(1 - bqx(1 - y^2)). \tag{3.18}$$

Continuing with inspiration from the half-plane solution [8], we try the Ansatz

$$S_w = \mu^w \sum_{n=0}^{\infty} c_n q^{-nw}. \tag{3.19}$$

For $n = 0$ we have

$$x(\mu^2 - \mu + y^2)c_0 = 0 \tag{3.20}$$

and for $n > 0$ we have

$$c_n = -\frac{\mu q^n c_{n-1}}{(y^2 q^{2n} - \mu q^n + \mu^2)qx}. \tag{3.21}$$

This implies that μ satisfies

$$\mu^2 - \mu + y^2 = 0 \tag{3.22}$$

for our Ansatz to work. We now parametrize y via

$$y^2 = \lambda(1 - \lambda) \tag{3.23}$$

which changes (3.20) into

$$x(\mu - \lambda)(\mu - 1 + \lambda)c_0 = 0 \tag{3.24}$$

and the recurrence into

$$c_n = -\frac{\mu q^n c_{n-1}}{(\mu - \lambda q^n)(\mu - (1 - \lambda)q^n)qx}. \tag{3.25}$$

We see immediately that either $\mu = \lambda$ or $\mu = 1 - \lambda$ and that we have two solutions from our Ansatz. This leads to the general solution for S_w as

$$S_w = A\lambda^w \sum_{n=0}^{\infty} \left(-\frac{1}{x}\right)^n q^{-nw} \prod_{k=0}^{n-1} \frac{q^k}{(1 - q^{k+1})(\lambda - (1 - \lambda)q^{k+1})} \\ +B(1 - \lambda)^w \sum_{n=0}^{\infty} \left(-\frac{1}{x}\right)^n q^{-nw} \prod_{k=0}^{n-1} \frac{q^k}{(1 - q^{k+1})((1 - \lambda) - \lambda q^{k+1})}, \tag{3.26}$$

and so for R_w via (3.16). One can then use the initial conditions (3.15) to solve for the coefficients A and B.

Defining

$$\phi_w^{(N)}(\rho, q) = \sum_{n=0}^{N} \frac{\rho^n q^{(n-N-w+1)n}}{(q;q)_n (q;q)_{N-n} (\rho q;q)_n (\frac{q}{\rho};q)_{N-n}}, \tag{3.27}$$

where

$$(t;q)_n = \prod_{k=0}^{n-1} (1 - tq^k) \tag{3.28}$$

is the standard q-product, after some lengthy calculations one finds

$$Q_w(x, y, q, a, 1) = q^{w(w+1)/2}\left[(1 - ax)T_w^{(1)}(x, q, \lambda) + (1 - qx - ax(1 - y^2))T_w^{(2)}(x, q, \lambda)\right] \tag{3.29}$$

with

$$T_w^{(1)}(x,q,\lambda) = \frac{1}{1-2\lambda}\sum_{N=0}^{w} q^{(N^2-3N)/2}(-x)^{w-N} \times\left[\frac{\lambda^w}{(1-\lambda)^{N-1}}\phi_w^{(N)}\left(\frac{1-\lambda}{\lambda},q\right) - \frac{(1-\lambda)^w}{\lambda^{N-1}}\phi_w^{(N)}\left(\frac{\lambda}{1-\lambda},q\right)\right] \tag{3.30}$$

and

$$T_w^{(2)}(x,q,\lambda) = -\frac{1}{1-2\lambda}\sum_{N=0}^{w} q^{(N^2-N-1)/2}(-x)^{w-1-N} \times\left[\frac{\lambda^w}{(1-\lambda)^{N}}\phi_{w+1}^{(N)}\left(\frac{1-\lambda}{\lambda},q\right) - \frac{(1-\lambda)^w}{\lambda^{N}}\phi_{w+1}^{(N)}\left(\frac{\lambda}{1-\lambda},q\right)\right]. \tag{3.31}$$

To obtain an expression for G_w one can then substitute (3.29) into (3.8) to obtain an expression for $Q_w(x,y,q,a,b)$ and then this into (3.10) to give $G_w(x,y,q,a,b)$.

4. The infinite width limit

For any finite length the partition function in the slit becomes equal to the partition function in the half-plane for large enough widths w. This means that the generating functions G_w approach G^{hp}, when they converge. The half-plane solution can be found in [8]. Here we take the limit $w \to \infty$ of the solution derived for finite width. This gives us a more compact expression for the denominator of the G^{hp} than appearing in [8].

After some work we find, using $\hat{\lambda} = 1-\lambda$,

$$Q^{\text{hp}}(x,y,q,a) = (1-ax)[P^{(1)}(x,q,\lambda)+P^{(2)}(x,q,\lambda)] + (1-qx-ax(1-y^2))[P^{(3)}(x,q,\lambda)+P^{(4)}(x,q,\lambda)] \tag{4.1}$$

with

$$P^{(1)}(x,q,\lambda) = \sum_{M=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x)^M\lambda^{M+m}\hat{\lambda}^{1-m}q^{\frac{1}{2}M^2+Mm+m^2+\frac{1}{2}M-m}}{(\hat{\lambda}-\lambda)(q;q)_\infty(q;q)_m(\lambda q/\hat{\lambda};q)_m(\hat{\lambda}q/\lambda;q)_\infty}, \tag{4.2}$$

$$P^{(2)}(x,q,\lambda) = \sum_{M=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x)^M\lambda^{1-m}\hat{\lambda}^{m+M}q^{\frac{1}{2}M^2+Mm+m^2+\frac{1}{2}M-m}}{(\lambda-\hat{\lambda})(q;q)_\infty(q;q)_m(\lambda q/\hat{\lambda};q)_\infty(\hat{\lambda}q/\lambda;q)_m}, \tag{4.3}$$

$$P^{(3)}(x,q,\lambda) = \sum_{M=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x)^M\lambda^{M+m+1}\hat{\lambda}^{-m}q^{\frac{1}{2}M^2+Mm+m^2+\frac{3}{2}M+m}}{(\lambda-\hat{\lambda})(q;q)_\infty(q;q)_m(\lambda q/\hat{\lambda};q)_m(\hat{\lambda}q/\lambda;q)_\infty}, \tag{4.4}$$

and

$$P^{(4)}(x,q,\lambda) = \sum_{M=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-x)^M\lambda^{-m}\hat{\lambda}^{M+m+1}q^{\frac{1}{2}M^2+Mm+m^2+\frac{1}{2}M-m}}{(\hat{\lambda}-\lambda)(q;q)_\infty(q;q)_m(\lambda q/\hat{\lambda};q)_m(\hat{\lambda}q/\lambda;q)_\infty}. \tag{4.5}$$

As in [8], we then find

$$G^{\text{hp}}(x,y,q,a) = \frac{Q^{\text{hp}}(x,y,q,0)}{Q^{\text{hp}}(x,y,q,a)}. \tag{4.6}$$

While this expression is equivalent to that found in [8], we note that its structure is fundamentally different.

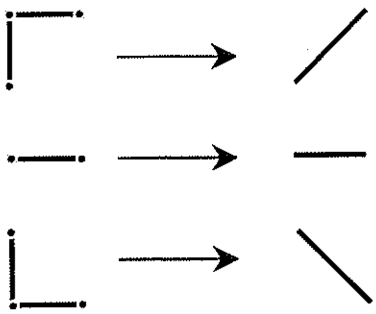

Figure 3. The mapping from RSOS paths to Motzkin paths: a horizontal step preceded by a vertical up-step is mapped to a Motzkin path up-step, a horizontal step with no preceding vertical step is mapped to a horizontal step and a horizontal step preceded by a vertical down-step is mapped to a Motzkin path down-step.

5. Mapping to Motzkin paths polymer model

As foreshadowed in the introduction, there is a mapping between RSOS configurations and Motzkin paths. Except for the zero-step configuration, every RSOS configuration in our definition (see section 2) ends with a horizontal step, and hence contains at least one horizontal step. The horizontal steps of a configuration can be uniquely partitioned into three sets as follows. Each horizontal step in an RSOS configuration is preceded by either a vertical step up, a vertical step down, or no vertical step. (In the latter case the horizontal step is either the first step, or it is immediately preceded by another horizontal step.) As there are no consecutive vertical steps in an RSOS configuration, this partitioning associates each of the vertical steps with the immediately following horizontal steps.

In this way, each non-zero length RSOS configuration is partitioned into vertical/horizontal step pairs and the remaining horizontal steps, as described above. These three classes of objects are then mapped to the three classes of steps of a Motzkin path as indicated in figure 3. A horizontal step preceded by a vertical up-step is mapped to a Motzkin path up-step, a horizontal step with no preceding vertical step is mapped to a horizontal step and a horizontal step preceded by a vertical down-step is mapped to a Motzkin path down-step. For example, the Motzkin path resulting from the mapping of the RSOS configuration in figure 1 can be seen in figure 4. Finally, the zero-step RSOS configuration is mapped to the zero-step Motzkin path. The mapping is clearly invertible, and as every Motzkin path gets mapped to an RSOS configuration by the inverse mapping, it follows that the mapping is bijective.

Clearly, the number of horizontal steps in an RSOS configuration is the same as the total number of steps in the corresponding Motzkin path, the number of up-steps in an RSOS configuration is the same as the number of up-steps in the corresponding Motzkin path and the number of down-steps in an RSOS configuration is the same as the number of down-steps in the corresponding Motzkin path.

The mapping is also area-preserving, as can be seen by the following argument. In an RSOS configuration there are equal numbers of up-steps and down-steps. When mapping an RSOS configuration to a Motzkin path, the area is decreased by half a unit square for every up-step, and it is increased by the same amount for every down-step. Therefore, the total area remains unchanged by the mapping.

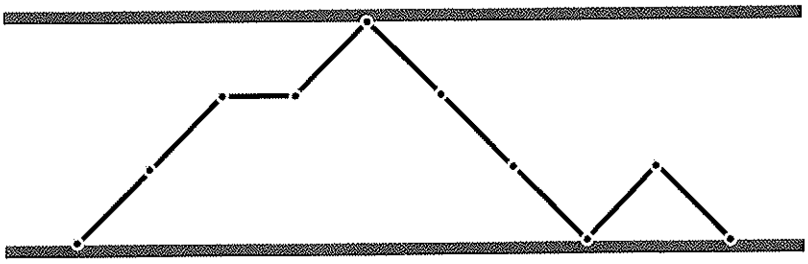

Figure 4. The mapping of RSOS configuration in figure 1 to a Motzkin path.

Moreover, for any RSOS configuration the number of edges in either surface is equivalent to the number of vertices of the corresponding Motzkin path in the corresponding surface, excluding the initial vertex of the Motzkin path. The mapping therefore preserves the number of contacts with either surface.

This implies that the generating function we have found is also that of Motzkin paths in a slit weighted by the area (which can be used to model an osmotic pressure [5]) interacting with both walls. In this way the current work is a generalization of that found in [7].

In this paper we have presented a solution to the linear RSOS model in a slit with field and wall interaction terms in the energy. In particular we have evaluated the generating function and demonstrated that its limit is the half-plane solution found earlier. The numerator and denominator polynomials of the slit generating function are novel q-orthogonal polynomials associated with the continued fraction expansion of the half-plane solution.

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