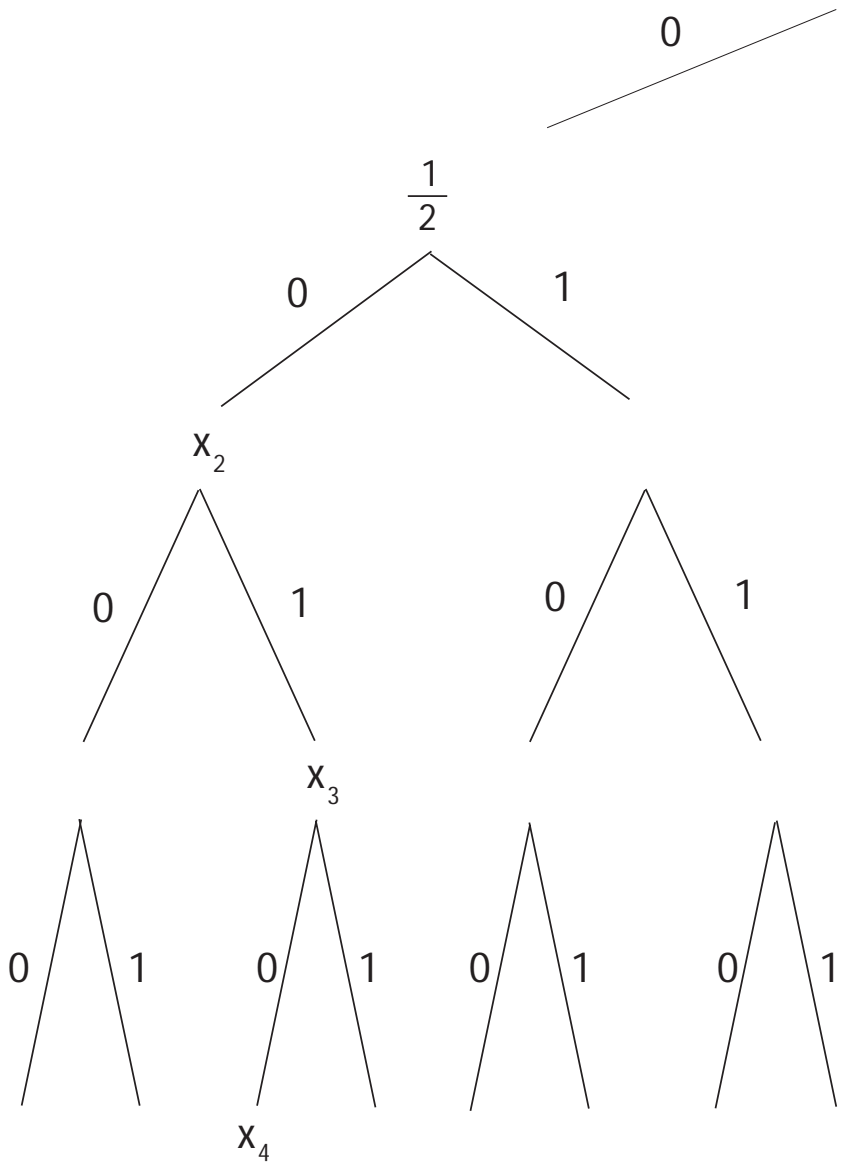


Maps, Spectra and Trees

Stefano Isola (Università di Camerino)

joint work with Andreas Knauf (Universität Erlangen)
and Mirko Degli Esposti (Università di Bologna)



Problem: Construct a 'dynamical' binary tree such that each $x \in [0, 1]$ can be uniquely approached along a finite or infinite path $\{x_k\}_{k \geq 1}$ with $x_1 = 1/2$.

For $k \geq 1$ the k -th row has 2^{k-1} elements (leaves) which can be enumerated lexicographically as follows: let $\mathbf{G}_k := (\mathbb{Z}/2\mathbb{Z})^k$ and define on for each $k \geq 1$ the involution

$$S : \mathbf{G}_k \rightarrow \mathbf{G}_k, \quad S(\sigma) = \bar{\sigma}$$

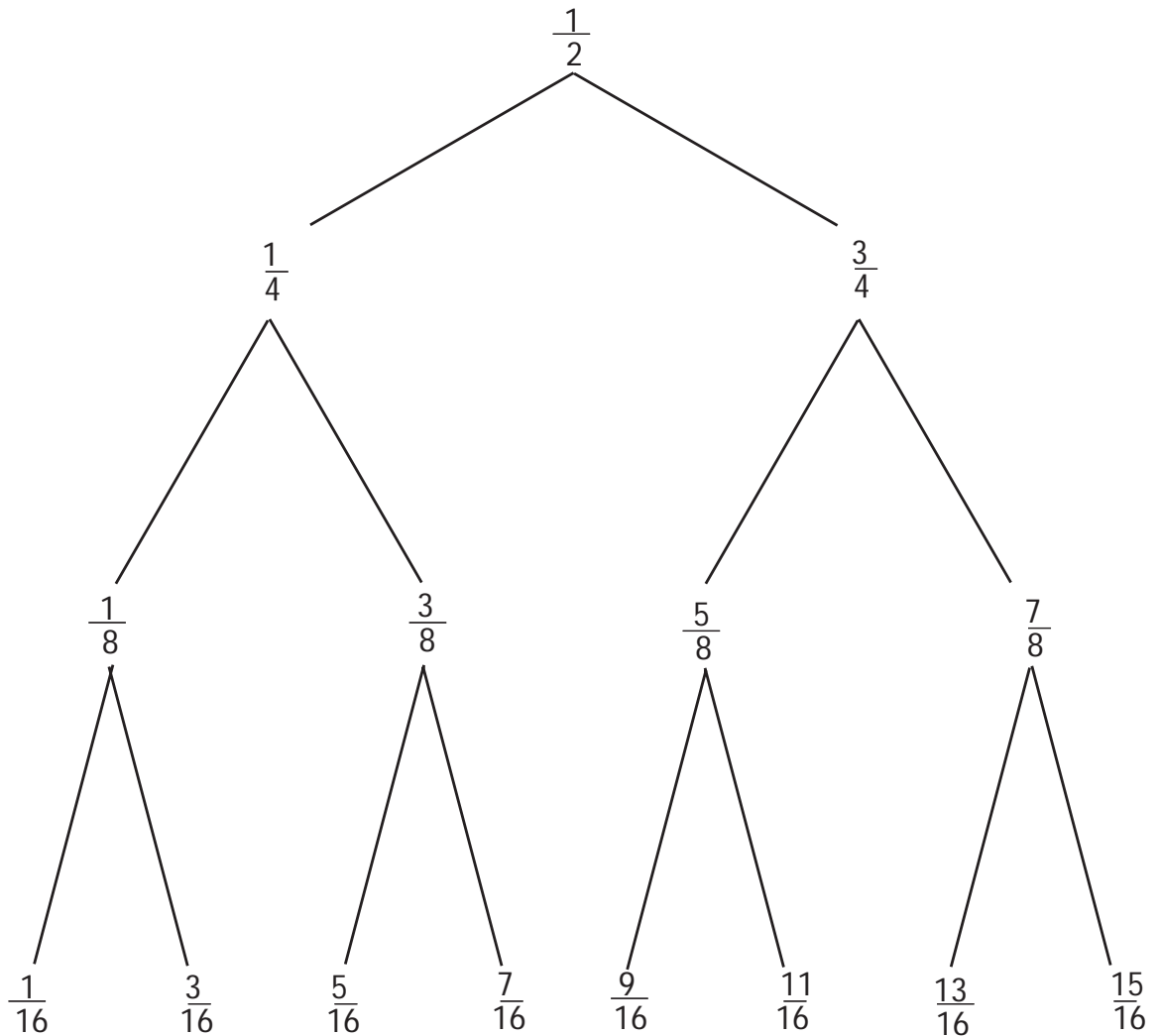
where $\bar{\sigma}_i = 1 - \sigma_i$. The quotient $\tilde{\mathbf{G}}_k := \mathbf{G}_k/S$ is again a group isomorphic to \mathbf{G}_{k-1} when for each equivalence class we choose the element starting with 0.

Dyadic tree \mathcal{D} : for each $\sigma \in \tilde{\mathbf{G}}_k$ set

$$x_k(\sigma) = 0.\sigma'1, \quad 0\sigma' = \sigma.$$

The leaves of \mathcal{D} are all dyadic rationals and the path on \mathcal{D} which converges to a given $x \in [0, 1]$ is the sequence of successive truncations of its binary expansion:

$$x = \sum_{i \geq 1} \sigma_i 2^{-i} \Rightarrow x_k = \sum_{i=1}^{k-1} \sigma_i 2^{-i} + 2^{-k}$$



Coding: To every $x \in [0, 1]$ with dyadic expansion $x = 0.\sigma$ there corresponds a unique sequence $\phi_0(x) \in \{0, 1\}^{\mathbb{N}}$ given by $\phi_0(x) = 0\sigma$ (replacing $1 \rightarrow 01^\infty$ if x is a dyadic rational) which represents an infinite path on \mathcal{D} , and viceversa.

Farey tree \mathcal{F}

Let $\sigma \in \tilde{\mathbf{G}}_k$ be of the form

$$\sigma = (\underbrace{0, \dots, 0}_{a_1}, \underbrace{1, \dots, 1}_{a_2}, \underbrace{0, \dots, 0}_{a_3}, \dots, \underbrace{u, \dots, u}_{a_{n-1}}, \underbrace{\bar{u}, \dots, \bar{u}}_{r-1})$$

with $u = 1$ ($u = 0$) for n odd (even), and integers $a_i > 0$ and $r > 1$, such that $k = \sum_{i=1}^{n-1} a_i + r - 1$.

Set

$$x_k(\sigma) = [a_1, a_2, \dots, a_{n-1} + \frac{1}{r}].$$

Alternatively, the k -th row of \mathcal{F} can be defined as the set $\mathcal{F}_k \setminus \mathcal{F}_{k-1}$ where \mathcal{F}_k (the k -th *modified Farey sequence*) is the ascending sequence of irreducible fractions between 0 and 1 constructed inductively from $\mathcal{F}_0 = (\frac{0}{1}, \frac{1}{1})$ by inserting mediants:

$$\mathcal{F}_1 = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right), \quad \mathcal{F}_2 = \left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right),$$

$$\mathcal{F}_3 = \left(\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right)$$

and so on.

The leaves of \mathcal{F} are all rationals and the path which converges to a given $x = [a_1, a_2, a_3, \dots]$ is the sequence of Farey convergents (FC's) yielding the *slow continued fraction algorithm*:

$$x_k = (k + 1)^{-1}, \quad k < a_1,$$

$$x_k = \frac{r p_{n-1} + p_{n-2}}{r q_{n-1} + q_{n-2}}, \quad \begin{cases} 1 \leq r \leq a_n, \\ k = \sum_{i=1}^{n-1} a_i + r - 1 \geq a_1. \end{cases}$$

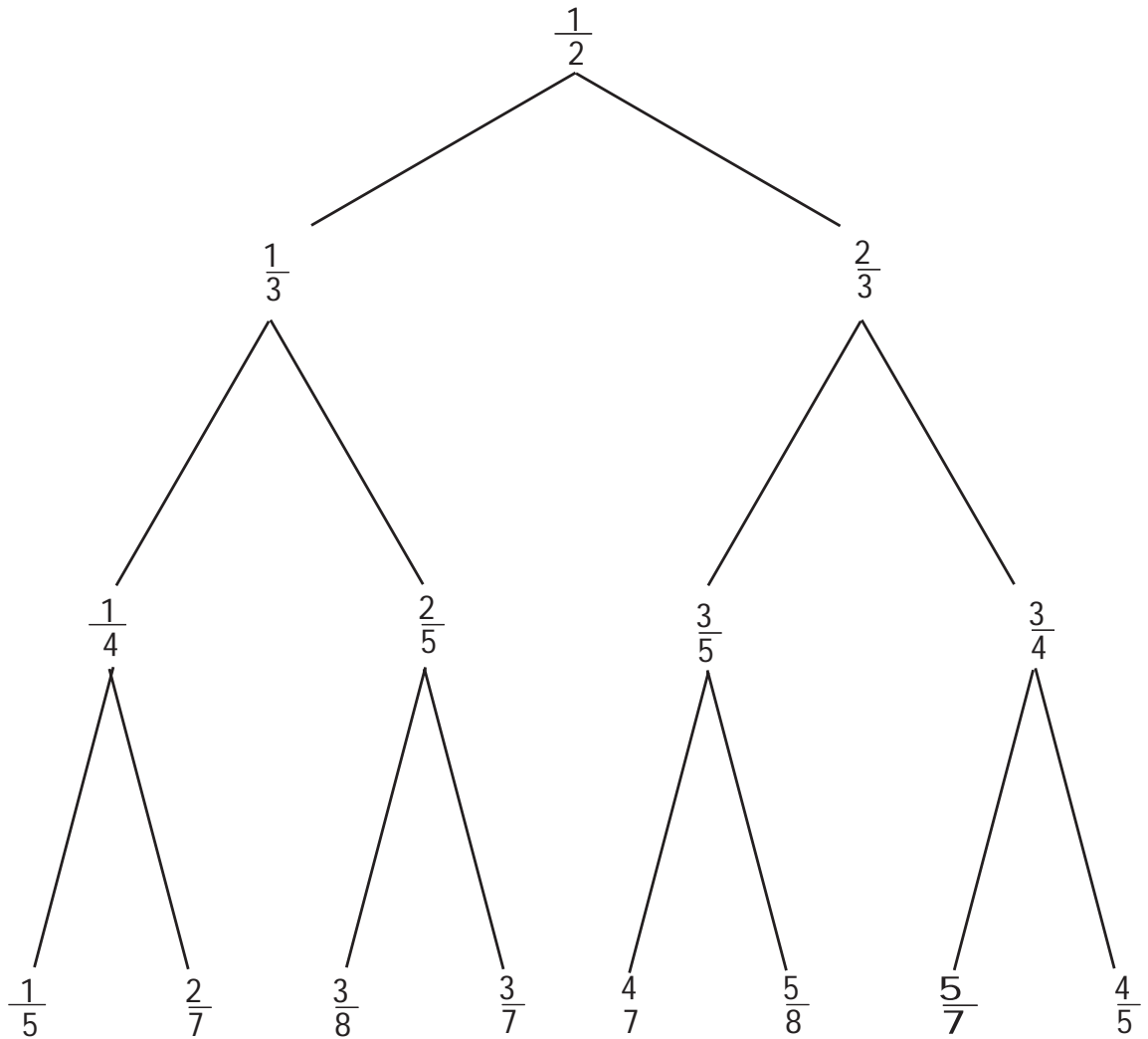
If $r = a_n$ then $x_k = p_n/q_n$, an ordinary continued fraction convergent (CFC), with:

$$\frac{p_0}{q_0} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{a_1}$$

and

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}, \quad n \geq 2.$$

The fraction $t_k/s_k := x_k$ is the *best one-sided rational approximation* to x whose denominator does not exceed s_k (although, if $r < a_n$, there might be a CFC with denominator less than s_k and closer to x on the other side of x).



Coding: To every $x \in [0, 1]$ with continued fraction expansion $x = [a_1, a_2, a_3, \dots]$ there corresponds a unique sequence $\phi_1(x) \in \{0, 1\}^{\mathbb{N}}$ given by $\phi_1(x) = 0^{a_1} 1^{a_2} 0^{a_3} \dots$ which represents an infinite path on \mathcal{F} (extended with 01^∞ or 10^∞ for rational x 's) along the sequence of FC's of x , and viceversa.

Growth of the denominators

CFC's denominators q_n typically grow exponentially fast:

$$\frac{\log q_n}{n} \rightarrow \frac{\pi^2}{12 \log 2} \quad \text{almost everywhere}$$

On the other hand, setting $x_k = t_k/s_k$ we have $\min\{s_k\} = k + 1$ whereas $\max\{s_k\} = (k + 1)$ -st Fibonacci number.

For all $k = \sum_{i=1}^{n-1} a_i + r - 1 \geq a_1$ it holds $q_{n-1} < s_k \leq q_n$. Moreover (Khinchin and Lévy):

$$\frac{1}{n \log n} \sum_{i=1}^n a_i \rightarrow \frac{1}{\log 2} \quad \text{in measure.}$$

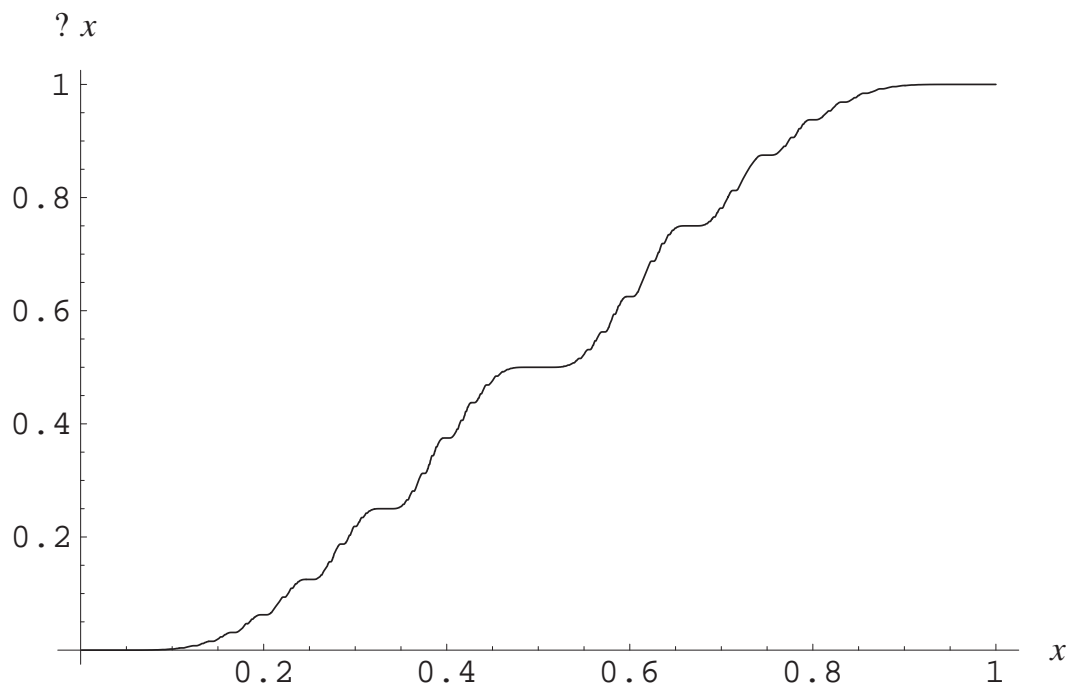
Therefore

$$\frac{\log s_k}{k} \sim \frac{\pi^2}{12 \log k} \quad \text{in measure.}$$

The Minkowski question mark

Given $x \in (0, 1)$ with continued fraction expansion $x = [a_1, a_2, a_3, \dots]$, what is the number obtained by interpreting the sequence $\phi_1(x)$ as the binary expansion of a real number in $(0, 1)$, i.e. what is $\phi_0^{-1} \circ \phi_1(x)$? The number so obtained is denoted $?(x)$ and writes

$$\begin{aligned} ?(x) &:= \sum_{k \geq 1} (-1)^{k-1} 2^{-(a_1 + \dots + a_k - 1)} \\ &= 0.\underbrace{00\dots 0}_{a_1-1} \underbrace{11\dots 1}_{a_2} \underbrace{00\dots 0}_{a_3} \dots \end{aligned}$$



Some properties:

- $\varphi(x)$ is strictly increasing from 0 to 1 and Hölder continuous of order $\beta = \frac{\log 2}{\sqrt{5}+1}$;
- x is rational iff $\varphi(x)$ is of the form $k/2^s$, with k and s integers;
- x is a quadratic irrational iff $\varphi(x)$ is a (non-dyadic) rational;
- $\varphi(x)$ is a singular function: its derivative vanishes Lebesgue-almost everywhere;
- it satisfies the functional eq. $\varphi(1 - x) = 1 - \varphi(x)$.

? maps the Farey tree \mathcal{F} to the dyadic tree \mathcal{D} :

Theorem

Since

$$x = \lim_{k \rightarrow \infty} \frac{\#\{\frac{p}{q} \in \mathcal{D}_k \setminus \{0\} : \frac{p}{q} \leq x\}}{2^k}$$

then

$$?(x) = \lim_{k \rightarrow \infty} \frac{\#\{\frac{p}{q} \in \mathcal{F}_k \setminus \{0\} : \frac{p}{q} \leq x\}}{2^k}.$$

Corollary

Let

$$c_n = \int_0^1 e^{2\pi i n x} d?(x)$$

then

$$c_n = \lim_{k \rightarrow \infty} \frac{1}{2^k} \sum_{\frac{p}{q} \in \mathcal{F}_k \setminus \{0\}} e^{2\pi i n \frac{p}{q}}.$$

The Farey map and the tent map

Let $F : [0, 1] \rightarrow [0, 1]$ and $T : [0, 1] \rightarrow [0, 1]$ be given by

$$F(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1-x}{x}, & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

and

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{2}, \\ 2(1-x), & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

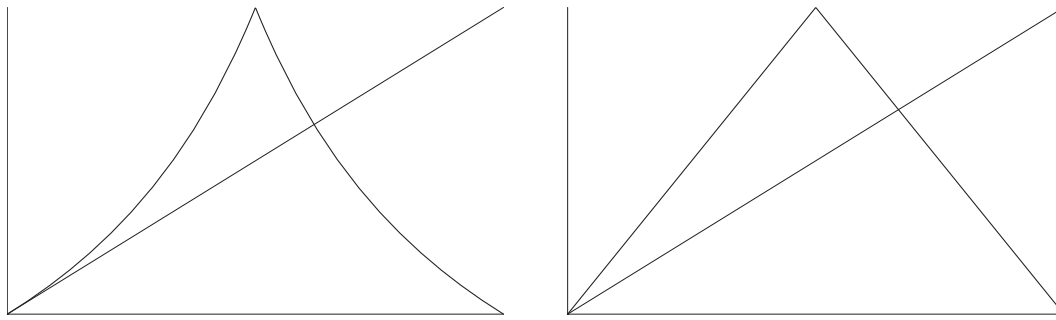
respectively. Then

$$\bigcup_{i=0}^{k+1} F^{-i}\{0\} = \mathcal{F}_k \quad \text{and} \quad \bigcup_{i=0}^{k+1} T^{-i}\{0\} = \mathcal{D}_k$$

and the k -th rows of the Farey and the dyadic tree are

$$F^{-(k-1)}\left(\frac{1}{2}\right) \quad \text{and} \quad T^{-(k-1)}\left(\frac{1}{2}\right)$$

respectively.



Theorem

$$\phi_1 \circ F \circ \phi_1^{-1} = \phi_0 \circ T \circ \phi_0^{-1}$$

and acts as the left-shift on $\Sigma := \{0, 1\}^{\mathbb{N}}/S$.

In particular,

$$\begin{array}{ccc} [0, 1] & \xrightarrow{F} & [0, 1] \\ \downarrow? & & \downarrow? \\ [0, 1] & \xrightarrow{T} & [0, 1] \end{array}$$

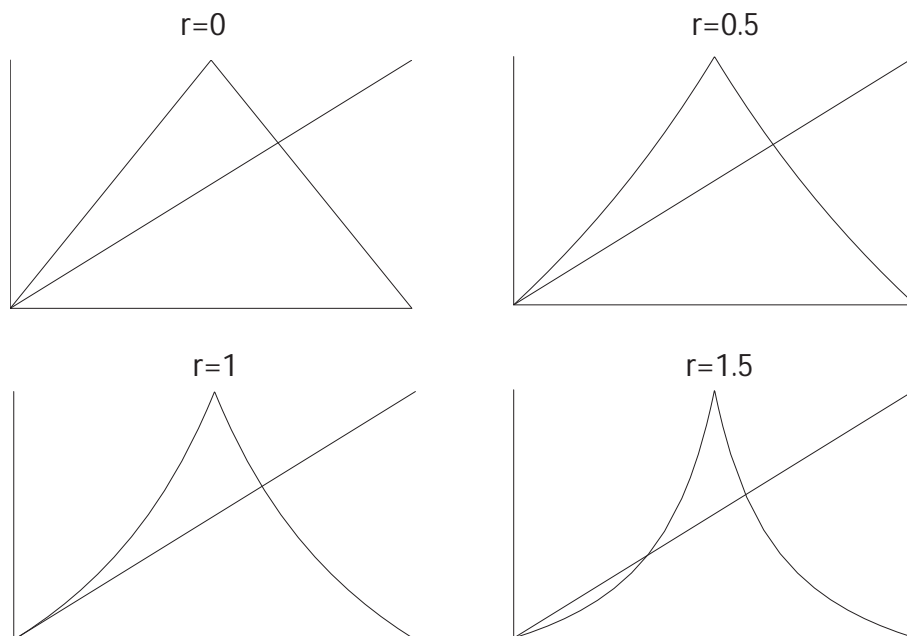
The measure $d^?(x)$ is F -invariant and its entropy is equal to $\log 2$ (this makes $d^?(x)$ the measure of maximal entropy for F). Being zero at every rational point $d^?$ is singular w.r.t. Lebesgue. On the other hand, F has an absolutely continuous infinite invariant measure with density $1/x$.

A one-parameter analytic Markov family

$$F_r(x) = \begin{cases} \frac{(2-r)x}{1-rx}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{(2-r)(1-x)}{1-r+rx}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

For $r \in [0, 2)$, $\inf |F'_r(x)| = F'_r(0) = 2 - r := \rho$.

Invariant density: $h_r(x) = (1 - r + rx)^{-1}$.



A one-parameter family of binary trees

For each $r \in [0, 2)$ one can construct as before a 'dynamical' binary tree $\mathcal{T}(r)$ from the sequences

$$\mathcal{T}_k(r) := \cup_{i=0}^{k+1} F_r^{-i}(0).$$

The ordered elements of $\mathcal{T}_k(r)$ can be written as ratios of irreducible polynomials over \mathbb{Z} .

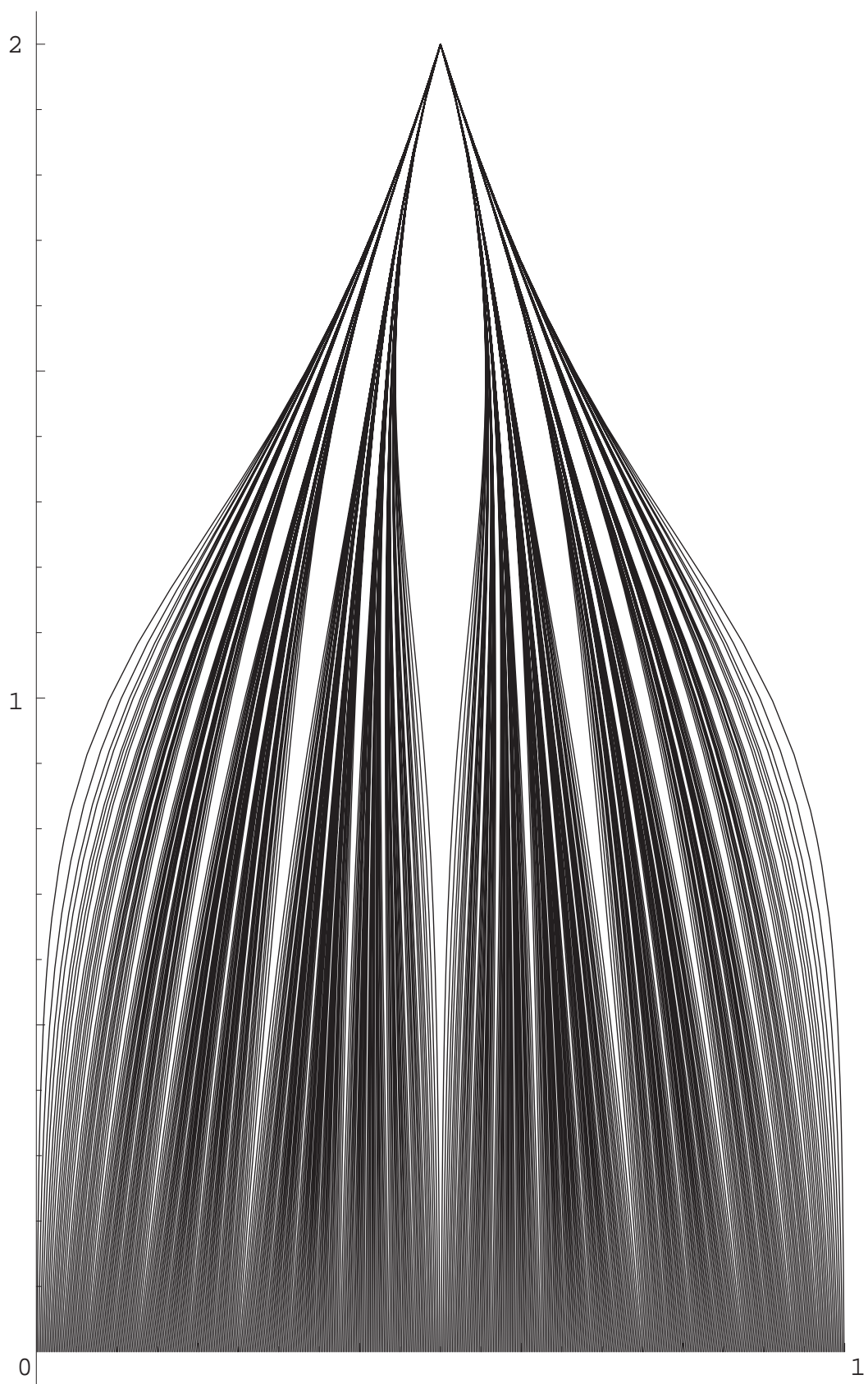
For example

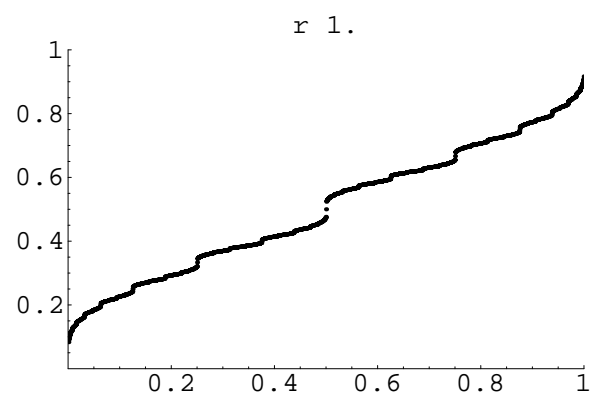
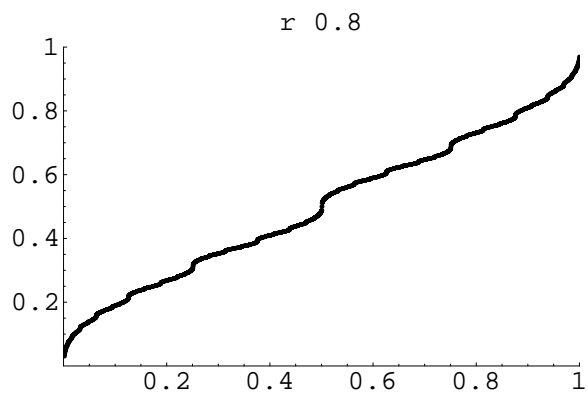
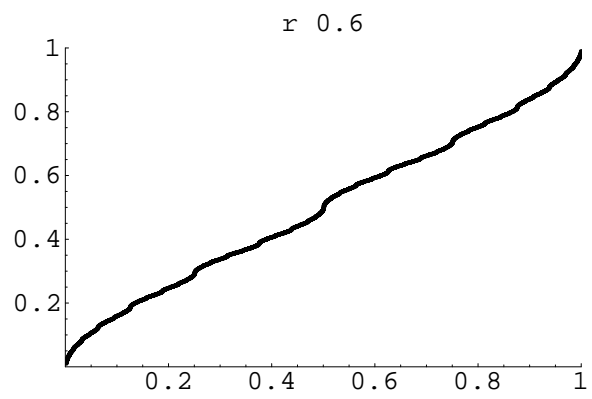
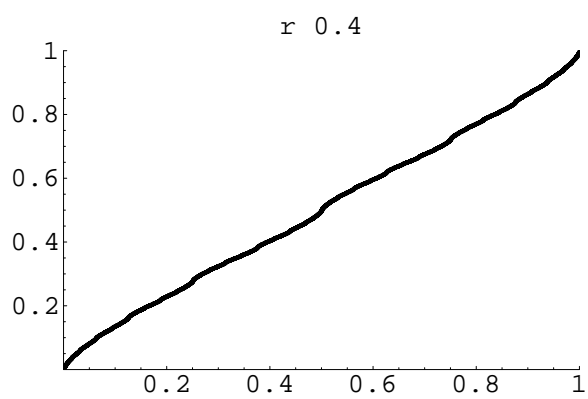
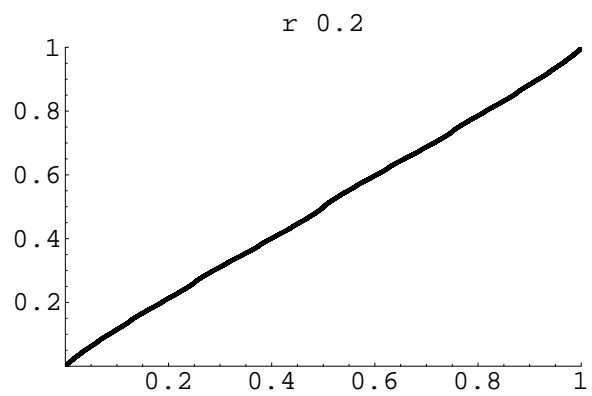
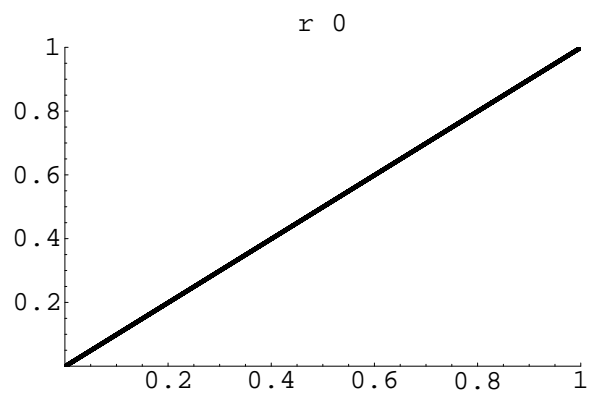
$$\mathcal{T}_0 = \left(\frac{0}{1}, \frac{1}{1} \right), \quad \mathcal{T}_1 = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right),$$

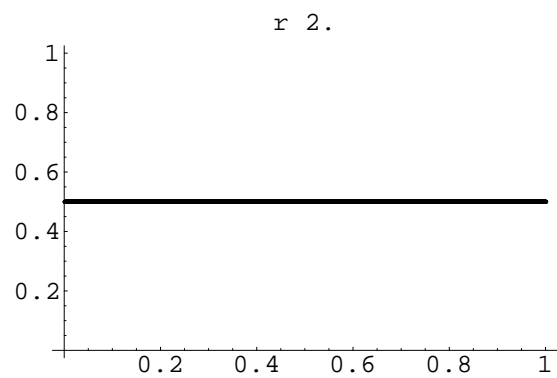
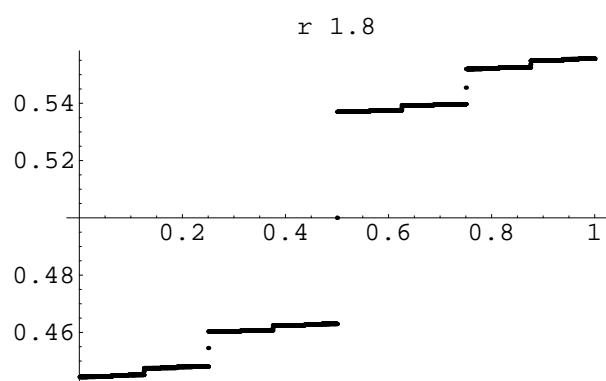
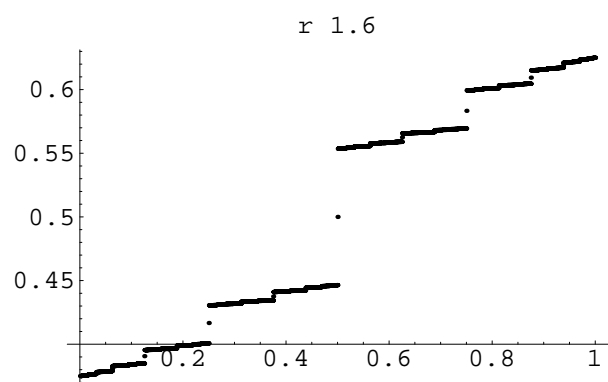
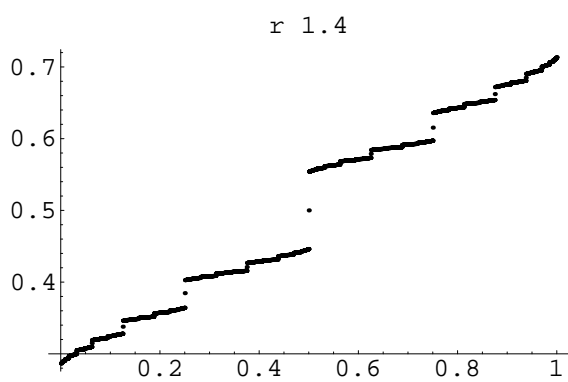
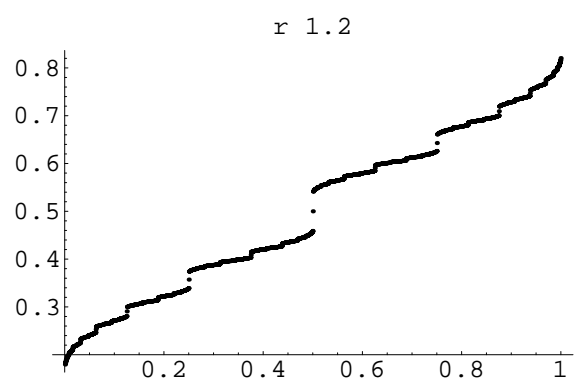
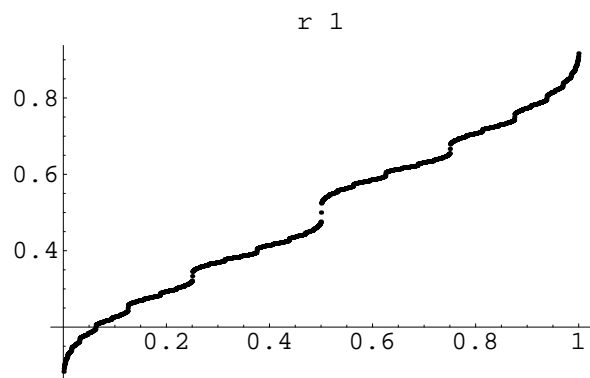
$$\mathcal{T}_2 = \left(\frac{0}{1}, \frac{1}{4-r}, \frac{1}{2}, \frac{3-r}{4-r}, \frac{1}{1} \right),$$

$$\mathcal{T}_3 \setminus \mathcal{T}_2 = \left(\frac{1}{r^2 - 5r + 8}, \frac{3-r}{8-3r}, \frac{5-2r}{8-3r}, \frac{r^2 - 5r + 7}{r^2 - 5r + 8} \right)$$

and so on.







Spin chains

The set $\mathcal{T}_k(r) \setminus \{0\} = \cup_{i=0}^{k+1} F_r^{-i}(1)$ contains 2^k elements of the form p_k/q_k which can be labelled with the elements of the group $\mathbf{G}_k = (\mathbb{Z}/2\mathbb{Z})^k$. Each $\sigma \in \mathbf{G}_k$ can then be interpreted as the configuration of k classical binary spins, with *energy function*

$$H_k = \log q_k : \mathbf{G}_k \rightarrow \mathbb{R}$$

The Fourier coefficients

$$j_k(\tau) = -2^{-k} \sum_{\sigma \in \mathbf{G}_k} H_k(\sigma) \cdot \chi_\tau(\sigma)$$

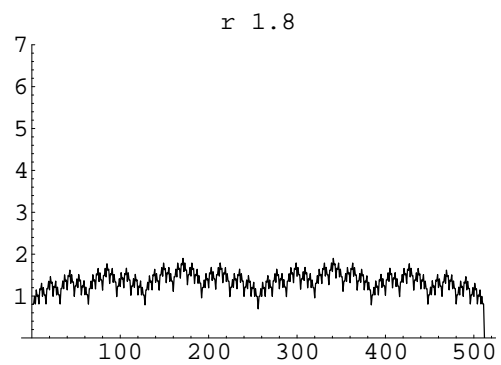
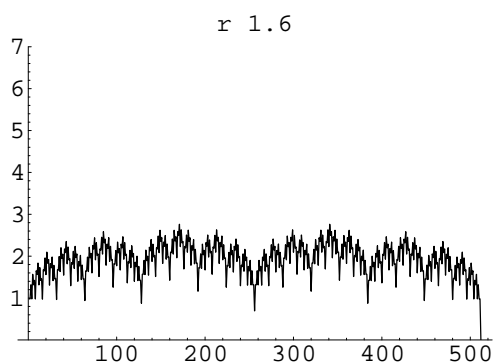
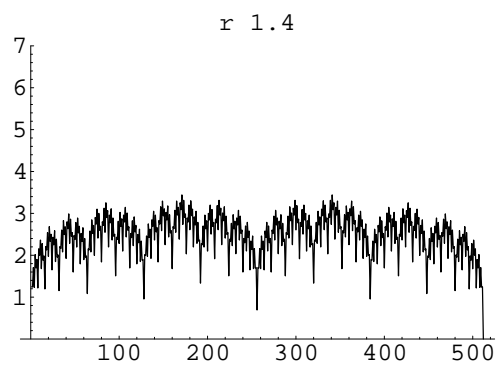
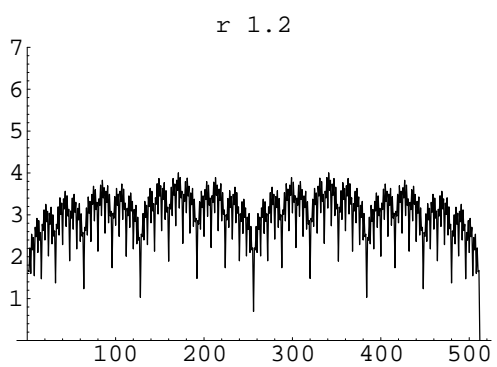
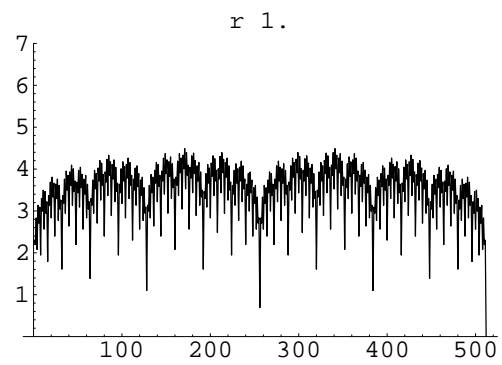
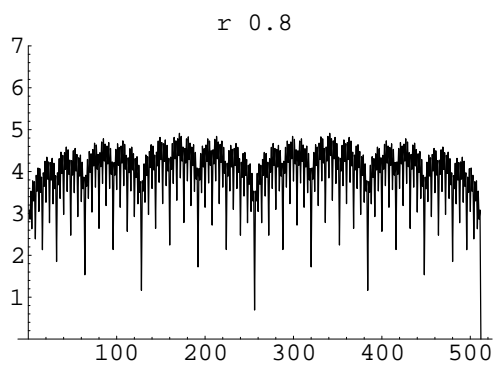
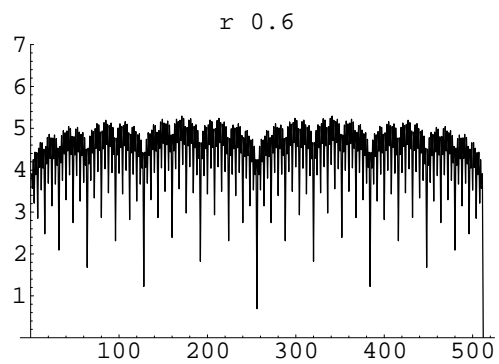
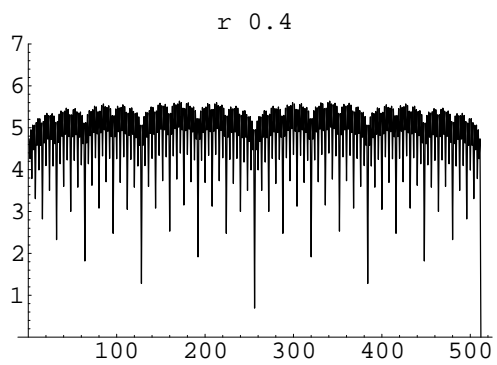
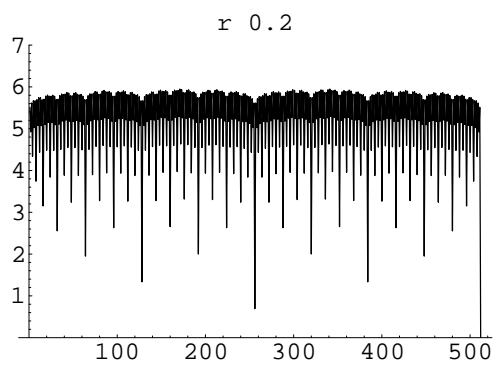
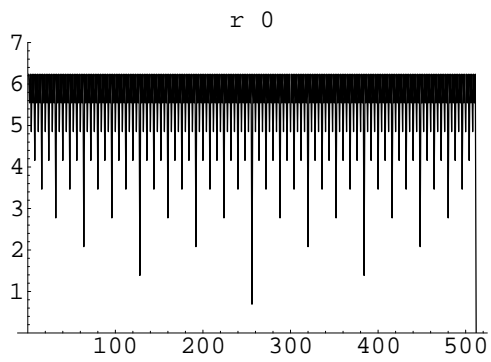
of $-H_k$, where $\chi_\tau(\sigma) = (-1)^{\sigma \cdot \tau}$ ($\tau \in \mathbf{G}_k^*$) are the characters on \mathbf{G}_k , are called *interaction coefficients* and

$$H_k(\sigma) = - \sum_{\tau \in \mathbf{G}_k^*} j_k(\tau) \cdot \chi_\sigma(\tau)$$

Theorem

The interaction is ferromagnetic for $r \in [0, 2)$:

$$j_k(\tau) \geq 0 \quad (\tau \in \mathbf{G}_k^* \setminus \{0\}).$$



The (canonical) partition function:

$$Z_k(\beta) = \sum_{\sigma \in \mathbf{G}_k} q_k(\sigma)^{-\beta} \equiv \sum_{\frac{p}{q} \in \mathcal{T}_k(r) \setminus \{0\}} q^{-\beta}$$

Example: $r = 0$

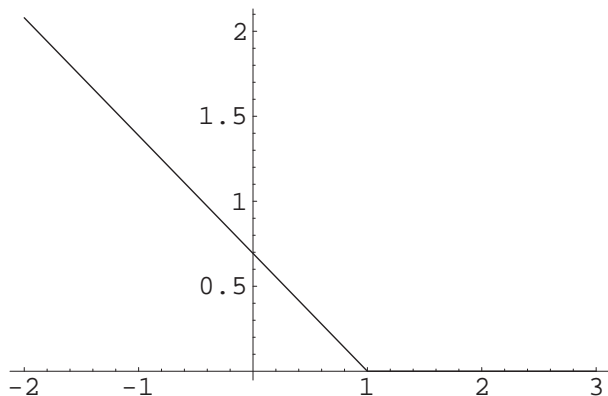
$$Z_k(\beta) = \frac{2^\beta - 1 - 2^{k(1-\beta)}}{2^\beta - 2}$$

so that

$$\lim_{k \rightarrow \infty} Z_k(\beta) = \frac{2^\beta - 1}{2^\beta - 2} = \frac{\zeta_0(\beta - 1)}{\zeta_0(\beta)}, \quad \text{Re}(\beta) > 1,$$

with $\zeta_0(\beta) = 2^\beta / (2^\beta - 1)$. The free energy is

$$-\beta f(\beta) = \lim_{k \rightarrow \infty} \frac{1}{k} \log Z_k(\beta) = \begin{cases} (1 - \beta) \log 2, & \beta < 1 \\ 0, & \beta \geq 1 \end{cases}$$



Example: $r = 1$ (Knauf's model)

$$\lim_{k \rightarrow \infty} Z_k(\beta) = \frac{\zeta(\beta - 1)}{\zeta(\beta)}, \quad \operatorname{Re}(\beta) > 2,$$

and

$$Z_k(2) \sim \frac{k}{2 \log k}, \quad k \rightarrow \infty.$$

The free energy $-\beta f(\beta)$ is real analytic for $\beta < 2$ and (Prellberg)

$$-\beta f(\beta) \sim \frac{2 - \beta}{-\log(2 - \beta)} \quad \text{as } \beta \rightarrow 2^-$$

Explicit values for $\beta = -k$, $k \in \mathbb{N}$:

$$\begin{aligned} -f(-1) &= \log 3 \\ -2f(-2) &= \log \left(\frac{5 + \sqrt{17}}{2} \right) \\ -3f(-3) &= \log 7 \\ -4f(-4) &= \log \left(\frac{11 + \sqrt{113}}{2} \right) \\ &\text{etc} \end{aligned}$$

Transfer operators: Given $r \in [0, 2)$, $\beta \in \mathbb{C}$ and $f : [0, 1] \rightarrow \mathbb{C}$ let

$$\mathcal{P}_{\beta,r}f(x) = \frac{\rho^\beta}{(rx + \rho)^{2\beta}} \left[f(\Phi_{r,0}(x)) + f(\Phi_{r,1}(x)) \right]$$

with inverse maps ($\rho = 2 - r$)

$$\Phi_{r,0}(x) = \frac{x}{rx + \rho} \quad \text{and} \quad \Phi_{r,1}(x) = 1 - \frac{x}{rx + \rho}$$

Involutions: The matrix

$$S_r = \begin{pmatrix} r - 1 & \rho \\ r & 1 - r \end{pmatrix} \in PSL(2, \mathbb{R})$$

with $S_r^2 = \text{Id}$ and $\det S_r = -1$ acts on \mathbb{C} as the Möbius transformation

$$x \rightarrow \hat{S}_r(x) = \frac{(r - 1)x + \rho}{rx + \rho - 1}$$

and on functions as

$$f \rightarrow (\mathcal{I}_{\beta,r}f)(x) = \frac{1}{(rx + \rho - 1)^{2\beta}} f(\hat{S}_r(x))$$

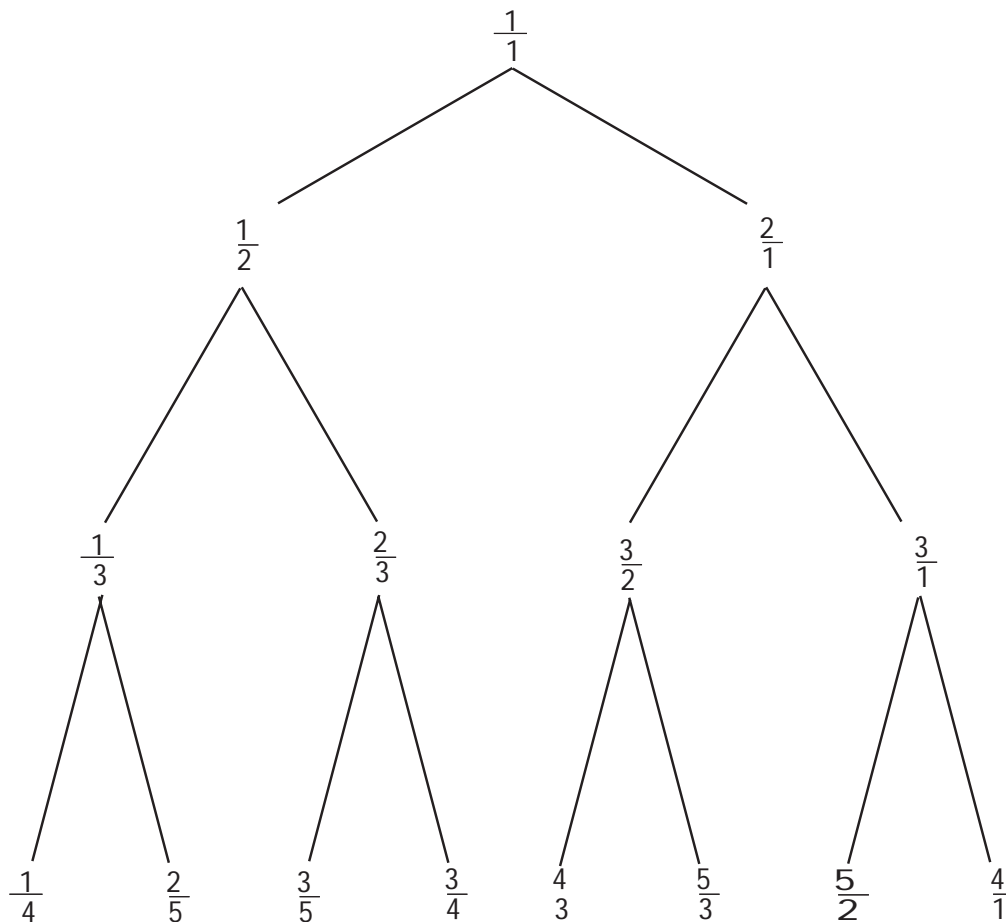
Since $\Phi_{r,i} \circ \hat{S}_r = \Phi_{r,1-i}$, $i = 0, 1$, we have

$$\mathcal{P}_{\beta,r}f = \lambda f, \quad \lambda \neq 0 \quad \iff \quad \mathcal{I}_{\beta,r}f = f$$

Extended trees: $\mathcal{T}(r)$ is a subtree of a larger tree having $\frac{1}{1}$ as root node (the 0-th row). Its k -th row R_k has 2^k leaves which once enumerated lexicographically with the group \mathbf{G}_k satisfy

$$x_k(\bar{\sigma}) = \hat{S}_r(x_k(\sigma))$$

For $r = 1$ this is the but the Stern-Brocot tree:



Theorem For all $r \in [0, 2)$, $\beta \in \mathbb{C}$, $k \geq 1$ and $f : [0, 1] \rightarrow \mathbb{C}$ we have

$$(\mathcal{P}_{\beta,r}^k f)(x) = \rho^{k\beta} \sum_{\frac{p}{q} \in R_k} \frac{f\left(\frac{n_0(x,p/q)}{prx+\rho q}\right) + f\left(\frac{n_1(x,p/q)}{prx+\rho q}\right)}{(prx + \rho q)^{2\beta}}$$

where the functions n_0 and n_1 can be computed recursively and satisfy:

$$n_0(x, p/q) + n_1(x, p/q) = prx + \rho q.$$

The choice $f \equiv 1$ and $x = 1$ yields

Corollary

$$2 Z_n(2\beta) = 1 + \sum_{k=0}^n \rho^{-k\beta} (\mathcal{P}_{\beta,r}^k 1)(1).$$

Thus, at least for $\beta \in \mathbb{R}$, $Z_n(\beta)$ has a finite limit as $n \rightarrow \infty$ whenever $\beta > \beta_{cr}$ where β_{cr} is twice the smallest positive real solution of the equation

$$\text{spec rad}(\mathcal{P}_{\beta,r}) = \rho^\beta$$

Remark 1

Note that (only) for $r = 1$ (due to arithmetical quibbles) we have

$$\mathcal{P}_{\beta,1}^n \mathbf{1}(0) = 1 + \sum_{k=0}^{n-1} \mathcal{P}_{\beta,1}^k \mathbf{1}(1)$$

and therefore

$$2 Z_{n-1}(2\beta) = \mathcal{P}_{\beta,1}^n \mathbf{1}(0)$$

This makes the ‘canonical’ and ‘grand canonical’ descriptions equivalent at all temperatures for $r = 1$. But for $r \neq 1$ this equivalence fails below β_{cr}^{-1} .

Remark 2

For $r = 1$ the phase transition at $\beta_{cr} = 2$ is of second order (although the magnetization jumps at β_{cr} from 1 to 0). On the other hand at $r = 0$ the first derivative of $-\beta f(\beta)$ is discontinuous (first order transition). This seems to be the general case, at least for $r \in [0, 1)$.

Generalizations: The choice $f \equiv 1$ can be generalized to $f(x) = e^{2\pi i m x}$, $m \in \mathbb{Z}$. Let

$$Z_n^{(m)}(\beta) := \sum_{\frac{p}{q} \in \mathcal{I}_n(r) \setminus \{0\}} q^{-\beta} e^{2\pi i m \frac{p}{q}}$$

then

$$2 Z_n^{(m)}(2\beta) = 1 + \sum_{k=0}^n \rho^{-k\beta} \mathcal{P}_{\beta,r}^k e^{2\pi i m x} \Big|_{x=1}.$$

The behaviour of the limit $\lim_{n \rightarrow \infty} Z_n^{(m)}(\beta)$ is related to the spectral properties of $\mathcal{P}_{\beta,r}$.

Example For $m = r = 1$ we have for $\operatorname{Re}(\beta) > 2$

$$\lim_{n \rightarrow \infty} Z_n^{(1)}(\beta) = \sum_{q \geq 1} \frac{\mu(q)}{q^\beta} = \frac{1}{\zeta(\beta)}$$

since the Möbius function

$$\mu\left(\prod p^{n_p}\right) = \begin{cases} (-1)^{\sum n_p}, & n_p \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies

$$\mu(q) = \sum_{\substack{0 < p \leq q \\ \gcd(p,q)=1}} e^{2\pi i \frac{p}{q}}, \quad q \in \mathbb{N}.$$

Spectral properties

Let \mathcal{H}_β the Hilbert space of all complex-valued functions f which can be represented as a generalized Borel transform

$$f(x) = (\mathcal{B}[\varphi])(x) := \frac{1}{x^{2\beta}} \int_0^\infty e^{-\frac{t}{x}} e^t \varphi(t) m_\beta(dt),$$

with $\varphi \in L^2(m_\beta)$ and $m_\beta(dt) = t^{2\beta-1} e^{-t} dt$.

Theorem

For all $r \in [0, 2)$ the space \mathcal{H}_β is invariant for $\mathcal{P}_{\beta,r}$, and

$$\mathcal{P}_{\beta,r} \mathcal{B}[\varphi] = \mathcal{B}[(M_{\beta,r} + N_{\beta,r})\varphi]$$

with

$$M_{\beta,r} \varphi(t) = \frac{e^{-\frac{r}{\rho}t}}{\rho^{2\beta-1}} \varphi\left(\frac{t}{\rho}\right)$$

and

$$N_{\beta,r} \varphi(t) = \frac{e^{\left(\frac{1-\rho}{\rho}\right)t}}{\rho^{2\beta-1}} \int_0^\infty \frac{J_{2\beta-1}\left(\frac{2\sqrt{st}}{\rho}\right)}{\left(\frac{st}{\rho}\right)^{\beta-1/2}} \varphi(s) m_\beta(ds).$$

Transition from discrete to continuous spectrum as $r \rightarrow 1^-$:

- For all $r \in [0, 1)$, the transfer operator $\mathcal{P}_{\beta,r}$ when acting on \mathcal{H}_β is of the trace-class, and

$$\operatorname{tr} \mathcal{P}_{\beta,r} = \frac{\rho^{1-\beta}}{r-1} + (4\rho)^{1-\beta} \frac{\sqrt{1+4\rho} - 1}{2\sqrt{1+4\rho}}$$

- For $r = 1$ $\mathcal{P}_{\beta,1}$ is self-adjoint in \mathcal{H}_β and its spectrum is the union of $[0, 1]$ and a (possibly empty) countable set of real eigenvalues of finite multiplicity.

Conjecture

For $\beta = r = 1$, $\mathcal{P}_{1,1} : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ has no eigenvalues $\neq 0$ and $\sigma(\mathcal{P}_{1,1}) = [0, 1]$.

General features of the eigenfunctions ($r = 1$): induced operators

Write $\mathcal{P}_\beta = \mathcal{P}_\beta^{(0)} + \mathcal{P}_\beta^{(1)}$ and for $z \in \mathbb{C}$ define

$$\mathcal{Q}_{\beta,z} = z \mathcal{P}_\beta^{(1)} (1 - z \mathcal{P}_\beta^{(0)})^{-1}$$

and

$$\mathcal{R}_{\beta,z} = z \mathcal{P}_\beta^{(0)} (1 - z \mathcal{P}_\beta^{(1)})^{-1}$$

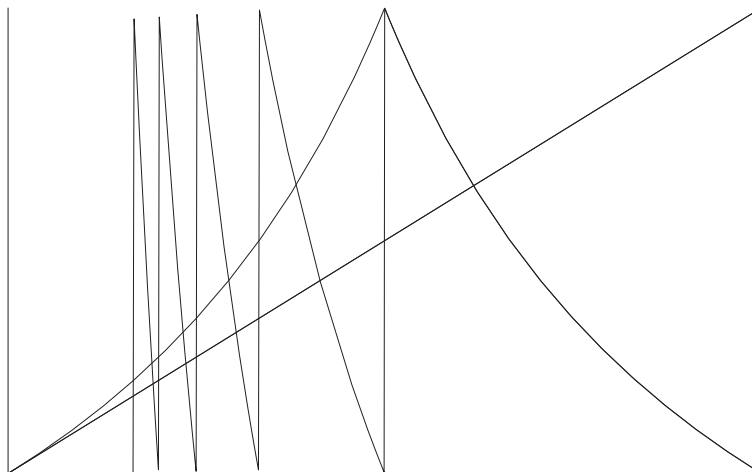
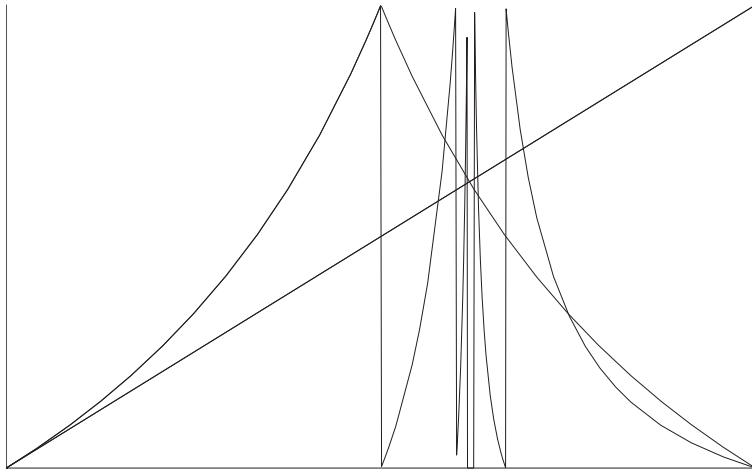
Power series expansions:

$$\mathcal{Q}_{\beta,z} f(x) = \sum_{n \geq 1} \frac{z^n}{(x+n)^{2\beta}} f\left(\frac{1}{x+n}\right)$$

and

$$\mathcal{R}_{\beta,z} f(x) = \sum_{n \geq 1} \frac{z^n}{(F_{n+1}x + F_n)^{2\beta}} f\left(\frac{F_n x + F_{n-1}}{F_{n+1}x + F_n}\right)$$

For bounded f absolute convergence in $\{|z| \leq 1, \operatorname{Re}(\beta) > 1/2\}$ and $|z| < \left(\frac{\sqrt{5}-1}{2}\right)^{-2\beta}$, respectively.



Invariant densities relation:

$$\frac{1}{x(x+1)} + \frac{1}{x+1} = \frac{1}{x}$$

Invariant spaces:

let \mathcal{K}_β be the Hilbert space of all complex-valued functions f which can be represented as a generalized Laplace transform of a function $\varphi \in L^2(m_\beta)$:

$$f(x) = (\mathcal{L}[\varphi])(x) := \int_0^\infty e^{-tx} \varphi(t) m_\beta(dt)$$

By Tricomi thm: $\mathcal{K}_\beta \subset \mathcal{H}_\beta$, with $\mathcal{L}[\varphi] = \mathcal{B}[N_\beta \varphi]$.

- $\mathcal{Q}_{\beta,z} : \mathcal{K}_\beta \rightarrow \mathcal{K}_\beta$ admits an analytic continuation in the cut plane $\mathbb{C} \setminus (1, \infty)$ and

$$\mathcal{Q}_{\beta,z} \mathcal{L}[\varphi] = \mathcal{L} [z(1 - zM)^{-1} N_\beta \varphi]$$

- $\mathcal{R}_{\beta,z} : \mathcal{H}_\beta \rightarrow \mathcal{H}_\beta$ can be meromorphically continued to \mathbb{C} with simple poles at $z = (-1)^{k-1} \left(\frac{\sqrt{5}-1}{2}\right)^{-2k\beta}$ ($k \geq 1$) and

$$\mathcal{R}_{\beta,z} \mathcal{B}[\varphi] = \mathcal{B} [z M(1 - zN_\beta)^{-1} \varphi]$$

Algebraic identity:

$$(1 - \mathcal{Q}_{\beta,z})(1 - z \mathcal{P}_{\beta}^{(0)}) = (1 - \mathcal{R}_{\beta,z})(1 - z \mathcal{P}_{\beta}^{(1)}) = 1 - z \mathcal{P}_{\beta}$$

Theorem

We have $\mathcal{P}_{\beta}f = \lambda f$ for some $f \in \mathcal{H}_{\beta}$ and $\lambda \notin \{0, 1\}$ if and only if f is analytic in $\operatorname{Re}(x) > 0$ and satisfies

$$f(x) = h_0(x) + h_1(x)$$

with $h_0 \in \mathcal{K}_{\beta}$ and $h_1 \in \mathcal{H}_{\beta}$ are such that

$$\mathcal{Q}_{\beta,1/\lambda}h_0 = h_0 \quad \text{and} \quad \mathcal{R}_{\beta,1/\lambda}h_1 = h_1$$

and satisfy

$$h_0 = \mathcal{I}_{\beta}h_1 \quad \text{and} \quad h_1 = \mathcal{I}_{\beta}h_0.$$

For $\lambda = 1$ the decomposition $f = h_0 + h_1$ reduces to the *Lewis functional equation*:

$$f(x) = f(x + 1) + x^{-2\beta}f\left(1 + \frac{1}{x}\right)$$

whereas $(1 - \mathcal{Q}_{\beta,1})h_0$ and $(1 - \mathcal{R}_{\beta,1})h_1$ are 1-periodic.