M. Sci. Examination by course unit Sample Exam Paper

MTH744U Dynamical Systems

Duration: 3 hours

Date and time:

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Examiner(s): T. Prellberg

Question 1 [36 marks]

For $r \in \mathbb{R}$, consider the differential equation

$$\dot{x} = rx - 2x^2 + x^3 \tag{1}$$

on the line.

(a) Show that $x^* = 0$ is a fixed point for any value of the parameter r, and determine its stability. Hence identify a bifurcation point r_1. [8 marks]

(b) Show that for certain values of the parameter r there are additional fixed points. For which values of r do these fixed points exist? Determine their stability and identify a further bifurcation point r_2. [12 marks]

(c) Using a Taylor expansion of (1), determine the normal form of the bifurcation at r_1. What type of bifurcation takes place. [6 marks]

(d) Similarly, determine the normal form of the bifurcation at r_2. What type of bifurcation takes place? [6 marks]

(e) Sketch the bifurcation diagram for all values of r and x^*. (Use a full line to denote a curve of stable fixed points, and a dashed line for a curve of unstable fixed points!) [4 marks]

Solution:

(a) We have $\dot{x} = f(x)$ with $f(x) = rx - 2x^2 + x^3$. For fixed points $f(x^*) = 0$, and clearly $f(0) = 0$, so $x^* = 0$ is a fixed point for all real r. [2]

We compute $f'(x) = r - 4x + 3x^2$. Hence $f'(0) = r$ and 0 is stable for $r < 0$ and unstable for $r > 0$. [4]

Hence a bifurcation takes place at $r_1 = 0$. [2]

(b) Additional fixed points are given by $r - 2x + x^2 = 0$, which implies that there are fixed points at $1 \pm \sqrt{1-r}$, provided $r \leq 1$. [4]

We compute $f'(1 \pm \sqrt{1-r}) = 2 - 2r \pm 2\sqrt{1-r} = 2\sqrt{1-r}(\sqrt{1-r} \pm 1)$. [3]

Hence $1 + \sqrt{1-r}$ is stable for $r < 1$, and $1 - \sqrt{1-r}$ is stable for $r < 0$ and unstable for $0 < r < 1$. [3]

Thus $r_2 = 1$ is another bifurcation point. [2]

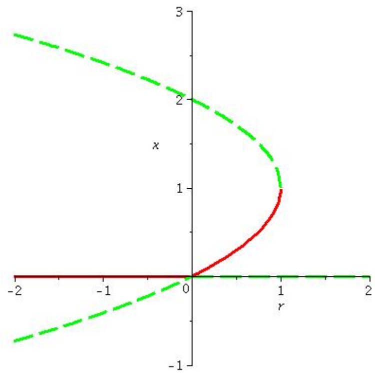
(c) Near $r = 0$ we expand to leading order $\dot{x} \approx rx - 2x^2$. The substitution $y = 2x$ gives $\dot{y} = ry - y^2$. [4]

This is the normal form of a transcritical bifurcation. [2]

(d) Near $r = 1$ we let $\tilde{r} = r - 1$ and expand to leading order in $y = x - 1$. We find $\dot{y} \approx \tilde{r} + \tilde{r}y + y^2$. Neglecting the term $\tilde{r}y$ we find $\dot{y} = r + y^2$. [4]

This is the normal form of a saddle-node bifurcation. [2]

(e) A sketch of the bifurcation diagram: [4]



Question 2 [34 marks]

Consider the dynamical system

$$\begin{aligned}\dot{x} =& x^2 - y - 1 \\ \dot{y} =& (x-2)y\end{aligned} \tag{2}$$

in the (x, y)-plane.

(a) Determine the nullclines of the system (2) and find the fixed points. [10 marks]

(b) Give the Jacobian matrix. Hence, determine the linear stability of all fixed points. [16 marks]

(c) Draw the nullclines of the system (2). Hence, sketch the phase portrait and the flow for the system (2). Indicate the stable and unstable manifolds for any saddles. [8 marks]

Solution:

(a) Note that $\dot{x} = 0$ implies $y = x^2 - 1$, and that $\dot{y} = 0$ implies $x = 2$ or $y = 0$. [2]

Therefore the parabola $y = x^2 - 1$ is the horizontal nullcline, and the line $x = 2$ and the x-axis are vertical nullclines. [2]

The horizontal and vertical nullclines intersect at the three fixed points $(2, 3)$, $(-1, 0)$ and $(1, 0)$. [6]

(b) The Jacobian matrix is $\begin{pmatrix} 2x & -1 \\ y & x-2 \end{pmatrix}$. [4]

At the fixed point $(2, 3)$ this reduces to $\begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}$, and thus the eigenvalues are 1 and 3. [2]

Hence the fixed point $(2, 3)$ is an unstable node. [2]

At the fixed point $(-1, 0)$ this reduces to $\begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$, and thus the eigenvalues are -2 and -3. [2]

Hence the fixed point $(-1, 0)$ is a stable node. [2]

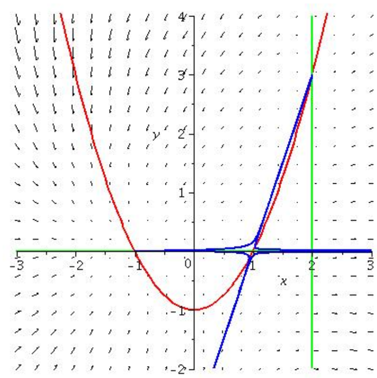
At the fixed point $(1,0)$ this reduces to $\begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}$, and thus the eigenvalues are 2 and -1. [2]

Hence the fixed point $(1,0)$ is a saddle. [2]

(c) A drawing of the nullclines [4]

indicating the flow directions [2]

and the stable and unstable manifolds of the saddle. [2]



Question 3 [30 marks]

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y - xy^2 \\ \dot{y} &= -x + yx^2 \end{aligned} \tag{3}$$

in the (x,y)-plane.

(a) Define a symmetry transformation R by $R(x,y) = (y,x)$. Show that the system is reversible (i.e. invariant under $(x,y) \to R(x,y)$, $t \to -t$). [6 marks]

(b) Show that there are invariant curves around the origin. [8 marks]

(c) Determine all fixed points of the system. [4 marks]

(d) Let $r^2 = x^2 + y^2$. Show that $\dot{r} = 0$. Considering the result of (c), what does this imply for the trajectories? [4 marks]

(e) Sketch the phase portrait in the (x,y)-plane, including trajectories through $(1,0)$ and $(2,0)$. Which fixed point (if any) does the trajectory through $(2,0)$ approach? [8 marks]

Solution:

(a) Under the transformation R, we find

$$\begin{aligned} \dot{y} &= x - yx^2 \\ \dot{x} &= -y + xy^2 \ , \end{aligned}$$

which is equivalent to the original system upon changing t to $-t$. [4]

Hence the system is reversible. [2]

(b) At the origin $\dot{x} = 0$ and $\dot{y} = 0$, hence the origin is a fixed point. [2]

The linearised map about this fixed point is $\dot{x} = y$ and $\dot{y} = -x$, which is a linear centre. [4]

As the system is reversible, the system has a non-linear centre at the origin. Therefore there are closed curves around the origin. [2]

(c) Fixed points simultaneously satisfy $0 = y(1 - xy)$ and $0 = -x(1 - xy)$. [2]

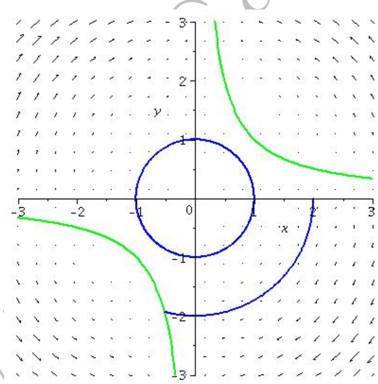
Hence in addition to the origin there is a line of fixed points on the hyperbola $y = 1/x$. [2]

(d) We compute directly $2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 0$. So the trajectories lie on circles around the origin. A trajectory therefore is either periodic or approaches a fixed point on the hyperbola $y = 1/x$. [4]

(e) The trajectory starting at $(1, 0)$ covers the whole circle of radius 1 in a clockwise fashion (it starts in direction $(0, -1)$. It is periodic. [2]

The trajectory starting at $(2, 0)$ moves along the circle of radius 2 in a clockwise fashion (it starts in direction $(0, -2)$). It approaches a fixed point for which $x^2 + 1/x^2 = 4$. We find $x = -\sqrt{2 - \sqrt{3}}$ and hence $y = -\sqrt{2 + \sqrt{3}}$. [2]

A sketch of phase portrait is as follows: [4]



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