Queen Mary University of London

M. Sci. Examination by course unit Sample Exam Paper

MTH744U Dynamical Systems

Duration: 3 hours

Date and time:

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): T. Prellberg

Question 1 [36 marks]

For $r \in \mathbb{R}$, consider the differential equation

$$\dot{x} = rx - 2x^2 + x^3 \tag{1}$$

on the line.

(a) Show that $x^* = 0$ is a fixed point for any value of the parameter r, and determine its stability. Hence identify a bifurcation point r_1. [8 marks]

(b) Show that for certain values of the parameter r there are additional fixed points. For which values of r do these fixed points exist? Determine their stability and identify a further bifurcation point r_2. [12 marks]

(c) Using a Taylor expansion of (1), determine the normal form of the bifurcation at r_1. What type of bifurcation takes place. [6 marks]

(d) Similarly, determine the normal form of the bifurcation at r_2. What type of bifurcation takes place? [6 marks]

(e) Sketch the bifurcation diagram for all values of r and x^*. (Use a full line to denote a curve of stable fixed points, and a dashed line for a curve of unstable fixed points!) [4 marks]

Question 2 [34 marks]

Consider the dynamical system

$$\begin{aligned} \dot{x} &= x^2 - y - 1 \\ \dot{y} &= (x-2)y \end{aligned} \tag{2}$$

in the (x, y)-plane.

(a) Determine the nullclines of the system (2) amd find the fixed points. [10 marks]

(b) Give the Jacobian matrix. Hence, determine the linear stability of all fixed points. [16 marks]

(c) Draw the nullclines of the system (2). Hence, sketch the phase portrait and the flow for the system (2). Indicate the stable and unstable manifolds for any saddles. [8 marks]

Question 3 [30 marks]

Consider the system of differential equations

$$\begin{aligned} \dot{x} =& y - xy^2 \\ \dot{y} =& -x + yx^2 \end{aligned} \tag{3}$$

in the (x, y)-plane.

(a) Define a symmetry transformation R by $R(x, y) = (y, x)$. Show that the system is reversible (i.e. invariant under $(x, y) \to R(x, y)$, $t \to -t$). [6 marks]

(b) Show that there are invariant curves around the origin. [8 marks]

(c) Determine all fixed points of the system. [4 marks]

(d) Let $r^2 = x^2 + y^2$. Show that $\dot{r} = 0$. Considering the result of (c), what does this imply for the trajectories? [4 marks]

(e) Sketch the phase portrait in the (x, y)-plane, including trajectories through $(1, 0)$ and $(2, 0)$. Which fixed point (if any) does the trajectory through $(2, 0)$ approach? [8 marks]

End of Paper