Queen Mary
University of London

M. Sci. Examination by course unit 2012

MTH744U Dynamical Systems

Duration: 3 hours

Date and time: 24th May 2012, 10.00am–1.00pm

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You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): T. Prellberg

Question 1 [38 marks]

For $r \in \mathbb{R}$, consider the differential equation

$$\dot{\theta} = r\cos(\theta) - \sin(2\theta)\ , \quad -\pi < \theta \leq \pi \tag{1}$$

on the circle.

(a) Show that $\theta^* = \pm\pi/2$ are fixed points for any value of the parameter r, and determine their stability. Hence, show that both of these fixed points undergo a bifurcation as r is varied. Find the bifurcation points $r_1 > 0$ and $r_2 < 0$ of the fixed points. [8 marks]

(b) Show that the bifurcations give rise to fixed points that satisfy

$$\sin(\theta^*) = \frac{r}{2}\ . \tag{2}$$

For which values of r do these fixed points exist? [6 marks]

(c) Determine the stability of the fixed points given by relation (2). [6 marks]

(d) Close to the bifurcation point r_1, Taylor expand relation (2) to second order in $\phi = \theta - \pi/2$ and derive an approximation for the fixed points as a function of r below the bifurcation point ($r \leq r_1$). [6 marks]

(e) Using a Taylor expansion of (1) in $\phi = \theta - \pi/2$, determine the normal form of the bifurcation at r_1. What type of bifurcation takes place? [6 marks]

(f) A similar bifurcation takes place at $r = r_2$. Sketch the bifurcation diagram for all values of r and θ^* (with $-\pi < \theta \leq \pi$). (Use a full line to denote stable fixed points, and a dashed line for unstable fixed points!) [6 marks]

Question 2 [37 marks]

For $r \in \mathbb{R}$, consider the dynamical system

$$\begin{aligned} \dot{x} =& xy^2 - rx \\ \dot{y} =& -xy - 2y \end{aligned} \tag{3}$$

in the (x, y)-plane.

(a) Find the fixed points of (3) as a function of r. Identify a bifurcation point and find the critical value r_c of the parameter at which the bifurcation occurs. (Make sure to carefully consider the case $r = r_c$.) [12 marks]

(b) Give the Jacobian matrix of the dynamical system (3). Hence, determine the linear stability of all fixed points for all values of $r \neq r_c$. [16 marks]

(c) Using the eigenvalues and eigenvectors of the linearised system, sketch the flow around the origin for $r < r_c$ and $r > r_c$ respectively. What happens near the origin at r_c (sketch and describe in words)? [9 marks]

Question 3 [25 marks]

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= -x - 2y^2 \\ \dot{y} &= xy - x^2 y \end{aligned} \tag{4}$$

in the (x, y)-plane.

(a) Determine the nullclines of the system (4) and show that the system (4) has only one fixed point (x^*, y^*). [8 marks]

(b) Show that $V(x, y) = (x - x^*)^2 + a(y - y^*)^2$ is a Lyapunov function for a suitably chosen value of a, and discuss the stability of the fixed point. [12 marks]

(c) Sketch the phase portrait of the system, clearly indicating nullclines and the direction of the flow in the regions separated by nullclines. Sketch a typical trajectory starting from a point $x_0 > 1, y_0 > 0$. [5 marks]

End of Paper