School of Mathematical Sciences
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Examiner: Dr T Prellberg

MTH5105 Differential and Integral Analysis
MID-TERM TEST

Date: 26-02-2010 *Time:* 12:10–12:50

Complete the following information:

Name	
Student Number (9 digit code)	

The test has THREE questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish to be marked.

Question	Marks
1	
2	
3	
Total Marks	

Nothing on this page will be marked!

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Question 1.

(a) State the formula for the Taylor polynomial $T_{n,a}$ of degree n of a function f at a, and state the Lagrange form of the remainder term R_n. [*10 marks*]

Let $f(x) = 1/\sqrt{1+x}$.

(b) Determine the Taylor polynomials $T_{2,0}$ and $T_{3,0}$ of degree 2 and 3, respectively, for f at $a = 0$. [*15 marks*]

(c) Using the Lagrange form of the remainder term, or otherwise, show that

$$T_{3,0}(x) < f(x) < T_{2,0}(x) \quad \text{for all } x > 0 \, .$$

[*10 marks*]

Answer 1.

(a)

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \text{and} \quad R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \, .$$

[*5 marks each*]

(b) From $f(x) = (1+x)^{-1/2}$ compute

$$f'(x) = -\frac{1}{2}(1+x)^{-3/2} \, , \quad f''(x) = \frac{3}{4}(1+x)^{-5/2} \, , \quad f'''(x) = -\frac{15}{8}(1+x)^{-7/2} \, .$$

Therefore $f(0) = 1$, $f'(0) = -1/2$, $f''(0) = 3/4$, $f'''(0) = -15/8$, and [*5 marks*]

$$T_{2,0}(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 \, , \quad \text{and} \quad T_{3,0}(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \, .$$

[*5 marks each*]

(c) $f(x) = T_{2,0}(x) + R_2$ for some $c \in (0, x)$, where

$$R_2 = \frac{-15/8(1+c)^{-7/2}}{3!}x^3 = -\frac{5}{16}\frac{x^3}{(1+c)^{7/2}} \, .$$

[*5 marks*]

As $0 < c < x$, we find

$$-\frac{5}{16}x^3 < R_2 < 0$$

and therefore $T_{3,0}(x) < f(x) < T_{2,0}(x)$. [*5 marks*]

Answer 1. (*Continue*)

Question 2.

(a) Give the definition of $f : \mathbb{R} \to \mathbb{R}$ being differentiable at a point $a \in \mathbb{R}$. [*10 marks*]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

(i) $f(x) = f(x-y)f(y)$ for all $x, y \in \mathbb{R}$, and

(ii) $f(x) - 1 = xg(x)$ with $\lim_{x\to 0} g(x) = 1$.

Show that f is differentiable and that $f'(a) = f(a)$ for all $a \in \mathbb{R}$. [*20 marks*]

Answer 2.

(a) f is differentiable at $a \in \mathbb{R}$ if the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. [*10 marks*]

(b) We compute

$$\begin{aligned}
f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} &= \lim_{x\to a} \frac{f(x-a)f(a) - f(a)}{x-a} && \text{[property (i)]}\\
= \lim_{x\to a} \frac{f(a)(f(x-a) - 1)}{x-a} &= f(a) \cdot \lim_{x\to a} \frac{f(x-a) - 1}{x-a} && \text{[limit laws]}\\
=f(a) \cdot \lim_{x\to a} \frac{(x-a)g(x-a)}{x-a} &= f(a) \cdot \lim_{x\to a} g(x-a) && \text{[introduce g]}\\
=f(a) \cdot 1 = f(a)\ . & && \text{[property (ii)]}
\end{aligned}$$

Hence the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is equal to $f'(a)$. [*5 marks for each line*]

Answer 2. (*Continue*)

Question 3.

(a) State Rolle's Theorem. [*15 marks*]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable with

$$f(0) = f(1) = f(2) = 0 \ .$$

Show that there exists a $c \in (0, 2)$ such that $f''(c) = 0$. [*20 marks*]

Answer 3.

(a) Rolle's Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b).

[*5 marks*]

If $f(b) = f(a)$ (also correct: if $f(b) = f(a) = 0$) then there is a $c \in (a, b)$

[*5 marks*]

such that

$$f'(c) = 0 \ .$$

[*5 marks*]

(b) (i) Apply Rolle's Theorem to f on $[0, 1]$:
f is (twice) differentiable on $\mathbb{R}$, hence continuous on $[0, 1]$ and differentiable on $(0, 1)$. Thus, there exists $a \in (0, 1)$ such that $f'(a)$=0.

[*5 marks*]

(ii) Apply Rolle's Theorem to f on $[1, 2]$:
f is (twice) differentiable on $\mathbb{R}$, hence continuous on $[1, 2]$ and differentiable on $(1, 2)$. Thus, there exists $b \in (1, 2)$ such that $f'(b)$=0.

[*5 marks*]

(iii) As $a \in (0, 1)$ and $b \in (1, 2)$, $a < b$ and we can apply Rolle's Theorem to f' on $[a, b]$:
f is twice differentiable on $\mathbb{R}$, hence f' is continuous on $[a, b]$ and differentiable on (a, b). Thus, there exists $c \in (a, b)$ such that $f''(c)$=0.

[*10 marks*]

Answer 3. (*Continue*)