MTH5105 Differential and Integral Analysis 2008-2009

Sample Test

Problem 1: (a) Give the definition of $f : \mathbb{R} \to \mathbb{R}$ being differentiable at a point $c \in \mathbb{R}$. [10 marks]

(b) Show directly from the definition that $f(x) = \frac{1}{x}$ is differentiable at any point c with $c \neq 0$, and find $f'(c)$. [10 marks]

(c) Suppose that f is continuous at 0. Show that the function g defined by $g(x) = xf(x)$ is differentiable at 0 at find its derivative. [10 marks]

Solution: (a) f is differentiable at $c \in \mathbb{R}$ if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists.

(b)

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{1/x - 1/c}{x - c} = \lim_{x \to c} \left(-\frac{1}{xc} \right) = -\frac{1}{c^2}$$

Thus f is differentiable at $c \neq 0$ and $f'(c) = -\frac{1}{c^2}$.

(c)

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{xf(x)}{x} = \lim_{x \to 0} f(x) = f(0)$$

since f is continuous at 0. Thus g is differentiable at 0 and $g'(0) = f(0)$.

Problem 2: (a) State Rolle's Theorem. [10 marks]

(b) State the Mean Value Theorem and prove it using Rolle's Theorem. [20 marks]

(c) Let f be continuous on $[a, b]$ and differentiable on (a, b). Suppose there exists a real constant M such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that for all $x, y \in [a, b]$

$$|f(x) - f(y)| \leq M|x - y|\,.$$

[10 marks]

Solution: (a) Rolle's Theorem: Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b) = 0$, then there is a $c \in (a, b)$ such that $f'(c) = 0$.

(b) MVT: Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

Proof: Let $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$. Then h satisfies the assumptions of Rolle's Theorem (h continuous on $[a, b]$ and differentiable on (a, b), $h(a) = h(b) = 0$) so that there is a $c \in (a, b)$ such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}\,.$$

(c) Fix $x, y \in [a, b]$.
Case 1: $x < y$. By the MVT there is a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}\,.$$

Hence $|f(y) - f(x)| = |f'(c)|\,|x - y| \leq M|x - y|$.
Case 2: $x > y$. Analogous to Case 1.
Case 3: $x = y$. $|f(x) - f(x)| \leq M|x - x|$ is true.

Problem 3: (a) Let $f(x) = \log(1+x)$.

(i) Determine the Taylor polynomials $T_{2,0}$ and $T_{3,0}$ of degree 2 and 3 at 0 for f. [15 marks]

(ii) Using the Lagrange form of the remainder, or otherwise, show that

$$T_{2,0}(x) \leq f(x) \leq T_{3,0}(x) \quad \text{for all } x \geq 0\,.$$

[15 marks]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Which of the following two statements (if any) is true?

(i) 'The Taylor series of f always converges for at least one point.'

(ii) 'The Taylor series of f always converges to the function for at least two points.'

[10 marks]

Solution: (a) $f(x) = \log(1+x)$, $f'(x) = 1/(1+x)$, $f''(x) = -1/(1+x)^2$, $f'''(x) = 2/(1+x)^3$.

(i) $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, and hence

$$T_{2,0}(x) = \frac{1}{1!}x + \frac{(-1)}{2!}x^2 = x - \frac{x^2}{2}\,,$$
$$T_{3,0}(x) = \frac{1}{1!}x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}\,.$$

(ii) For $x > 0$ there is a $c \in (0,x)$ such that

$$f(x) = T_{2,0}(x) + \frac{2/(1+c)^3}{3!}x^3\,.$$

But $0 < \frac{2/(1+c)^3}{3!}x^3 < \frac{x^3}{3}$, so that

$$T_{2,0}(x) < f(x) < T_{3,0}(x)\,.$$

For $x = 0$ we have $T_{2,0}(0) = f(0) = T_{3,0}(0)$, so that for all $x \geq 0$

$$T_{2,0}(x) \leq f(x) \leq T_{3,0}(x)\,.$$

(b) (i) True: the Taylor series always converges for $x = 0$.

(ii) False: the Taylor series of

$$f(x) = \begin{cases} 0 & x = 0 \\ \exp(-1/x^2) & x \neq 0 \end{cases}$$

does not converge to $f(x)$ for $x \neq 0$.

Thomas Prellberg, February 2009