

**B. Sc. Examination by course unit 2009**

**MTH5105 Differential and Integral Analysis**

**Duration: 2 hours**

**Date and time: sample paper showing the format of the exam.**

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Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.
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Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): T. Prellberg

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**Question 1** Suppose that  $f: [0, 1] \rightarrow \mathbb{R}$  is continuously differentiable.

- (a) Show that there is some number  $M$  such that  $|f'(x)| \leq M$  for all  $x$ .
- (b) Using the Mean Value Theorem, or otherwise, prove that

$$|f(x) - f(y)| \leq M|x - y|$$

for all  $x, y \in [0, 1]$ .

[You may use any theorems from the course provided that they are stated clearly.]

[25 marks]

**Question 2** (a) State the Fundamental Theorem of Calculus. Use it to prove the integration by parts formula

$$\int_a^b u(x)v'(x) dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x) dx.$$

You should state what conditions  $u$  and  $v$  should satisfy (e.g., continuity, differentiability etc.)

- (b) Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Using the chain rule, or otherwise, show that the function  $F$  defined by

$$F(x) = \exp\left(\int_0^x f(t) dt\right)$$

is differentiable and find its derivative.

[25 marks]

**Question 3** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \cos(x + \pi/3)$ . Find the Taylor Polynomial  $T_{3,0}(x)$  of degree three for  $f$ . Find the remainder term (in the Lagrange Form) and deduce that  $|f(x) - T_{3,0}(x)| \leq x^4/24$  for all  $x$ .

Hence, show that

$$\left| \int_0^1 f(x) dx - \int_0^1 T_{3,0}(x) dx \right| \leq 1/120.$$

[25 marks]

**Question 4** Let  $f: [0, \pi] \rightarrow \mathbb{R}$  be defined by  $f(x) = \sqrt{\sin x}$ .

- (a) Why do we know that  $f$  is integrable on  $[0, \pi]$ ?
- (b) By considering the lower sum for the partition  $x_0 = 0, x_1 = \pi/6, x_2 = 5\pi/6, x_3 = \pi$ , or otherwise, prove that  $\int_0^\pi f \geq \sqrt{2}\pi/3$ .
- (c) Is the function  $F(x) = \int_0^x f$  continuous? [Justification is not required]
- (d) Deduce that there is some number  $c \in (0, \pi)$  such that  $\int_0^c f = 1$ .

[25 marks]

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End of Paper