Queen Mary
University of London

B. Sc. Examination by course unit 2010

MTH5105 Differential and Integral Analysis

Duration: 2 hours

Date and time: 28 May 2010, 14.30–16.30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): T. Prellberg

Question 1 (a) Let $a, x \in \mathbb{R}$ with $a < x$. Let the real-valued function f be n times continuously differentiable on $[a, x]$ and $(n+1)$ times continuously differentiable on (a, x).

(i) Write down the n-th Taylor polynomial $T_{n,a}$ of f at a, and write down both integral and Lagrange forms of the remainder

$$R_{n,a} = f - T_{n,a} \ .$$

(ii) Find the Taylor polynomial $T_{2,1}$ of f at $a = 1$ for

$$f(x) = (1 + 2x)^{-1/2} \ ,$$

and find both integral and Lagrange forms of the remainder $R_{2,1}$.

(b) Let $g(x) = \log(1 - x)$.

(i) Write down the Taylor series at zero for g.

(ii) By factorising $1 - x^4$, or otherwise, determine the Taylor series at zero for $f(x) = \log(1 + x + x^2 + x^3)$ up to order x^7.

[25 marks]

Question 2 Suppose that the function $f : [0, 1] \to \mathbb{R}$ is decreasing.

(a) State why $\int_0^1 f(x)\, dx$ exists.

(b) Given the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ of $[0, 1]$, find the upper and lower sums $U(f, P_n)$ and $L(f, P_n)$.

(c) Let

$$S_n = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right) \ .$$

Prove that

$$S_n \leq \int_0^1 f(x)\, dx \leq S_n + \frac{1}{n}\left(f(0) - f(1)\right) \ .$$

Hence deduce that $S_n \to \int_0^1 f(x)\, dx$ as $n \to \infty$.

(d) By considering the function $f(x) = (2 + x)^{-2}$, prove that

$$n\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} + \ldots + \frac{1}{(3n)^2}\right) \to \frac{1}{6}$$

as $n \to \infty$.

[25 marks]

Question 3 We say that a real-valued function f defined on an interval I is *Lipschitz* if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all x and y in I.

(a) State the Boundedness Principle and the Mean Value Theorem.

(b) By using the Boundedness Principle and the Mean Value Theorem, or otherwise, prove that every continuously differentiable function on $[0, 1]$ is Lipschitz.

(c) What does it mean to say that a real-valued function f defined on an interval I is uniformly continuous?

(d) Show that a Lipschitz function is uniformly continuous.

[25 marks]

Question 4 For $m \in \mathbb{N}$, define $f_m : \mathbb{R} \to \mathbb{R}$ by

$$f_m(x) = \frac{x}{m^2 + x^2}\ .$$

(a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{m=1}^{\infty} f_m(x)$ converges.

(b) Show that the sum $\sum_{m=1}^{\infty} f'_m(x)$ converges uniformly for all $x \in \mathbb{R}$.
[Hint: $|m^2 - x^2| \leq m^2 + x^2$]

(c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{m=1}^{\infty} \frac{x}{m^2 + x^2}$$

is differentiable. What is $f'(x)$?

[25 marks]

End of Paper