Queen Mary
University of London

B. Sc. Examination by course unit 2009

MTH5105 Differential and Integral Analysis

Duration: 2 hours

Date and time: 13th May 2009, 10.00–12.00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): T. Prellberg

Question 1 Let $f : [0, \pi/2] \to \mathbb{R}$ be given by

$$f(x) = \int_0^x \sqrt{\cos t}\, dt \ .$$

(a) Determine $f'(x)$. Justify your answer.

(b) Find the Taylor polynomial $T_{2,0}(x)$ for f at zero.

(c) What is the remainder term R_2? Using the fact that $|f^{(3)}(c)| < 1$ for all $c \in (0,\, 1/10)$, show that $|R_2| < 1/6000$ for $x = 1/10$.

(d) Deduce that

$$1/10 - 1/6000 < \int_0^{1/10} \sqrt{\cos t}\, dt < 1/10 + 1/6000 \ .$$

[25 marks]

Question 2 Define $f : [-2, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \ . \end{cases}$$

(a) Prove that f is differentiable at the point $a = 0$ and determine $f'(0)$.

(b) Suppose P is a partition of $[1, 2]$.

(i) Show that $L(f, P) = 0$.

(ii) Show that $U(f, P) \geq 1$.

(iii) Deduce that $\int_1^2 f(x)\, dx$ does not exist.

(c) Does $\int_{-2}^2 f(x)\, dx$ exist?

[25 marks]

Question 3 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose that $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$, and that $a, b \in \mathbb{R}$.

(a) Using the Mean Value Theorem, or otherwise, prove that $|f(b) - f(a)| \leq |b - a|$.

(b) Using the previous part prove that f is uniformly continuous.

(c) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable but not uniformly continuous. *[Justification is not required.]*

[25 marks]

Question 4 (a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$ converges.
[You may use that $|\sin(t)| \leq |t|$ for all $t \in \mathbb{R}$.]

(b) Show that the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$ converges uniformly for all $x \in \mathbb{R}$.

(c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$$

is differentiable.

[25 marks]

End of Paper