MTH5105 Differential and Integral Analysis 2008-2009

Sample Test

Problem 1: (a) Give the definition of $f : \mathbb{R} \to \mathbb{R}$ being differentiable at a point $c \in \mathbb{R}$.

[10 marks]

(b) Show directly from the definition that $f(x) = \frac{1}{x}$ is differentiable at any point c with $c \neq 0$, and find $f'(c)$. [10 marks]

(c) Suppose that f is continuous at 0. Show that the function g defined by $g(x) = xf(x)$ is differentiable at 0 at find its derivative. [10 marks]

Problem 2: (a) State Rolle's Theorem. [10 marks]

(b) State the Mean Value Theorem and prove it using Rolle's Theorem.

[20 marks]

(c) Let f be continuous on $[a, b]$ and differentiable on (a, b). Suppose there exists a real constant M such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that for all $x, y \in [a, b]$

$$|f(x) - f(y)| \leq M|x - y| \,.$$

[10 marks]

Problem 3: (a) Let $f(x) = \log(1 + x)$.

(i) Determine the Taylor polynomials $T_{2,0}$ and $T_{3,0}$ of degree 2 and 3 at 0 for f. [15 marks]

(ii) Using the Lagrange form of the remainder, or otherwise, show that

$$T_{2,0}(x) \leq f(x) \leq T_{3,0}(x) \quad \text{for all } x \geq 0 \,.$$

[15 marks]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable. Which of the following two statements (if any) is true?

(i) 'The Taylor series of f always converges for at least one point.'

(ii) 'The Taylor series of f always converges to the function for at least two points.'

[10 marks]

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