7. Properties of the Riemann Integral

Theorem 35 Let $f:[a,b]\to\mathbb{R}$ be R-integrable.

If $[c,d]\subset[a,b]$ then f is R-integrable over $[c,d]$.

Proof Let $\varepsilon>0$. Then there is a $P\in\mathbb{P}$ with $U(f,P)-L(f,P)<\varepsilon$.

If we let $P'=P\cup\{c,d\}=\{x_0,x_1,\dots,x_k=c,x_{k+1},\dots,x_{k+r}=d,\dots,x_n\}$

then $$U(f,P')-L(f,P')\le U(f,P)-L(f,P)<\varepsilon$$

Now take $P''=\{x_k,x_{k+1},\dots,x_{k+r}\}$, which is a partition of $[c,d]$ with

$$U(f,P'')-L(f,P'')=\sum_{i=k+1}^{k+r}(M_i-m_i)\Delta x_i$$

$$\le\sum_{i=1}^{n}(M_i-m_i)\Delta x_i$$

$$=U(f,P')-L(f,P')<\varepsilon$$

Thus f is R-integrable over $[c,d]$. ▯

2 March 05

Theorem 36 Let $f:[a,b]\to\mathbb{R}$ be R-integrable over $[a,c]$ and $[c,b]$

where $a<c<b$. Then f is R-integrable over $[a,b]$ and

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$

Proof Let $\varepsilon > 0$ and P_1 a partition of $[a,c]$,

P_2 a partition of $[c,b]$ with

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$$

$$U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$$

Thus $P = P_1 \cup P_2$ is a partition of $[a,b]$ with

$$U(f,P) - L(f,P) = U(f,P_1) + U(f,P_2) - L(f,P_1) - L(f,P_2) < \varepsilon$$

and hence f is R-integrable on $[a,b]$. Moreover

$$L(f,P_1) \le \int_a^c f(x)\,dx \le U(f,P_1)$$

$$L(f,P_2) \le \int_c^b f(x)\,dx \le U(f,P_2)$$

so that $L(f,P) \le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \le U(f,P)$

By definition, $L(f,P) \le \int_a^b f(x)\,dx \le U(f,P)$

so that

$$-\varepsilon < L(f,P) - U(f,P) \le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx$$

$$\le U(f,P) - L(f,P) < \varepsilon$$

and thus $\forall \varepsilon > 0$ $\left| \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx \right| < \varepsilon$

Thus, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ $\Box$

Definition for $a > b$, we define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Theorem 37 Let $f, g : [a,b] \to \mathbb{R}$ be bounded and $P \in \mathcal{P}$.

Then (a) $U(f+g, P) \leq U(f,P) + U(g,P)$

(b) $L(f+g, P) \geq L(f,P) + L(g,P)$

Proof (a) On a subinterval I_i,

$$M_i(f+g) = \sup_{x \in I_i} \big(f(x) + g(x)\big) \leq \sup_{x \in I_i} f(x) + \sup_{x \in I_i} g(x)$$

$$= M_i(f) + M_i(g).$$

Thus $$U(f+g, P) = \sum_{i=1}^{n} M_i(f+g)\Delta x_i \leq \sum_{i=1}^{n} M_i(f)\Delta x_i + \sum_{i=1}^{n} M_i(g)\Delta x_i$$

$$= U(f,P) + U(g,P)$$

(b) analogous. $\square$

Theorem 38 Let $f, g : [a,b] \to \mathbb{R}$ be R-integrable and $c \in \mathbb{R}$.

Then $f+g$ and cf are R-integrable, and

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\,dx + \int_b^c g(x)\,dx$$

$$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$$

Proof (a) Let $\varepsilon > 0$. There exist partitions P_1, P_2 such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2} \qquad U(g,P_2) - L(g,P_1) < \frac{\varepsilon}{2}$$

and if $P = P_1 \cup P_2$ then

$$U(f,P) - L(f,P) \leq U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$$
$$U(g,P) - L(g,P) \leq U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$$

By Theorem 37,

$$U(f+g,P) - L(f+g,P) \leq U(f,P) + U(g,P) - L(f,P) - L(g,P) < \varepsilon$$

As in the proof of Theorem 36 we can show that also

$$\left| \int_a^b f(x)+g(x)\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < \varepsilon$$

(b) One can show that $U(cf, P) - L(cf,P)$

$$\leq |c| \big(U(f,P) - L(f,P)\big).$$

The rest is coursework.

Theorem 39 Let $f : [a,b] \to \mathbb{R}$ be R-integrable.

If $g : [a,b] \to \mathbb{R}$ differs from f at finitely many points

then g is also R-integrable with

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

Proof For $c \in [a,b]$, define $\chi_c(x) = \begin{cases} 1 & x = c \\ 0 & x \neq c \end{cases}$

Then $g(x) = f(x) + \sum_{i=1}^{n} (g(c_i) - f(c_i))\, \chi_{c_i}(x)$

and it suffices to show that $\chi_c(x)$ is R-integrable with $\int_a^b \chi_c(x)\,dx = 0$.

5 March 9

If $a < c < b$ then choose $P = \{a, x_1, x_2, b\}$ with $a < x_1 < c < x_2 < b$

and $x_2 - x_1 < \varepsilon$ to get $0 = L(\chi_c, P) < U(\chi_c, P) < \varepsilon$.

If $c = a$ then choose $P = \{a, x_1, b\}$ with $a < x_1 < b$ and $x_1 - a < \varepsilon$

to get $0 = L(\chi_a, P) < U(\chi_a, P) < \varepsilon$ (similar if $c = b$).

Thus for all $\varepsilon > 0$ there is a partition P with $U(\chi_c, P) - L(\chi_c, P) < \varepsilon$

So χ_c is R-integrable and $\int_a^b \chi_c(x)\,dx = 0$ $\square$

Theorem 40 Let $f, g : [a,b] \to \mathbb{R}$ be R-integrable. If

$f(x) \leq g(x)$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Proof $g(x) - f(x) \geq 0$, and thus $0 \leq L(g - f, P) \leq \int_a^b g(x) - f(x)\,dx$

$$= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \quad \square$$

<u>Theorem 41</u> If $f : [a,b] \to \mathbb{R}$ is R-integrable, then $|f|$ is R-integrable and $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

<u>Proof</u> For $P \in \mathcal{P}$, we define

$$M_i = \sup_{x \in I_i} f(x) \quad ; \quad M_i^* = \sup_{x \in I_i} |f(x)|$$

$$m_i = \inf_{x \in I_i} f(x) \quad , \quad m_i^* = \inf_{x \in I_i} |f(x)|$$

Now starting with
$$\big| |f(x)| - |f(y)| \big| \leq |f(x) - f(y)|$$
we can show $M_i^* - m_i^* \leq M_i - m_i$ (Coursework)

so that
$$U(|f|,P) - L(|f|,P) = \sum_{i=1}^{n} (M_i^* - m_i^*)\,\Delta x_i$$
$$\leq \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i = U(f,P) - L(f,P)$$

As f is R-integrable, it follows that $|f|$ is R-integrable.

From $-|f(x)| \leq f(x) \leq |f(x)|$

it now follows that
$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \qquad \square$$

Theorem 42 If $f:[a,b]\to\mathbb{R}$ is R-integrable then f^2 is R-integrable.

Proof As f is bounded, $|f(x)|\le M$ for all $x\in[a,b]$.

Given a partition P, we have $M_i(f^2)=(M_i(|f|))^2$

and $m_i(f^2)=(m_i(|f|))^2$

so that

$$M_i(f^2)-m_i(f^2)=\underbrace{\big(M_i(|f|)+m_i(|f|)\big)}_{\le\ 2M}\big(M_i(|f|)-m_i(|f|)\big)$$

$$\le 2M\,\big(M_i(|f|)-m_i(|f|)\big)$$

Thus $U(f^2,P)-L(f^2,P)\le 2M\big(U(|f|,P)-L(|f|,P)\big)$

Hence f^2 is R-integrable $\left(\text{and } \int_a^b f^2(x)\,dx \le 2M\int_a^b |f(x)|\,dx\right)$ $\square$

Theorem 43 If $f,g:[a,b]\to\mathbb{R}$ are R-integrable then fg is R-integrable

Proof $$f(x)\,g(x)=\frac{1}{4}\Big(\big(f(x)+g(x)\big)^2-\big(f(x)-g(x)\big)^2\Big)$$

$f+g$ and $f-g$ are R-integrable by Theorem 38,

and thus $(f+g)^2$ and $(f-g)^2$ are R-integrable by Theorem 42.

By Theorem 38 it follows that $fg=\frac{1}{4}(f+g)^2-\frac{1}{4}(f-g)^2$ is R-integrable. $\square$

8. The Fundamental Theorem of Calculus

Definition 44 Let I be an interval and $f: I \to \mathbb{R}$.

A differentiable function $F: I \to \mathbb{R}$ is called an antiderivative of f

if $F'(x) = f(x)$ for all $x \in I$

Theorem 45 If F and G are antiderivatives of f, then

$G = F + c$ for some $c \in \mathbb{R}$. Also, $F + c$ is an antiderivative for all $c \in \mathbb{R}$

Proof $(G-F)' = G' - F' = f - f = 0$, so $G - F$ is constant.

Also $(F+c)' = F' = f$ for all $c \in \mathbb{R}$ □

Theorem 46 (Fundamental Theorem of Calculus, FTC) Let $f: [a,b] \to \mathbb{R}$ be R-integrable. If F is an

antiderivative of f then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

Proof Let P be a partition of $[a,b]$. Applying the MVT

to I_i, there exists c_i s.t. $x_{i-1} < c_i < x_i$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x_i$$

We have $$m_i = \inf_{x \in I_i} f(x) \le f(c_i) \le \sup_{x \in I_i} f(x) = M_i$$

so that $$m_i \Delta x_i \le F(x_i) - F(x_{i-1}) \le M_i \Delta x_i$$

and thus $$L(f,P) \le \underbrace{\sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right)}_{F(b) - F(a)} \le U(f,P)$$

Therefore

$$\int_{a}^{b} f(x)\,dx \le F(b)-F(a) \le \int_{a}^{b} f(x)\,dx$$

and f is R-integrable, so that $\displaystyle\int_a^b f(x)\,dx = F(b)-F(a)$ $\square$

Examples $\displaystyle\int_1^a \frac{dx}{x} = \log x \Big|_1^a = \log a - \log 1 = \log a$ etc.

Theorem 47 Let $f:[a,b]\to\mathbb{R}$ be R-integrable and define $F:[a,b]\to\mathbb{R}$ by $\displaystyle F(t)=\int_a^t f(x)\,dx$.

Then (a) F is continuous on $[a,b]$

(b) If f is continuous at $c\in[a,b]$ then F is differentiable at c and $F'(c)=f(c)$.

Proof (a) f is R-integrable, hence bounded, i.e. $|f(x)|\le M$ for all $x\in[a,b]$.

For $t, t_0\in[a,b]$ we have

$$|F(t)-F(t_0)| = \left|\int_a^t f(x)\,dx - \int_a^{t_0} f(x)\,dx\right| = \left|\int_{t_0}^{t} f(x)\,dx\right| \le M\,|t-t_0|$$

if $|t-t_0|<\delta=\frac{\varepsilon}{M}$ then $|F(t)-F(t_0)|<\varepsilon$, hence F is continuous.

(b) $\forall \varepsilon>0\ \exists\delta>0: |x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$.

Thus $\forall t: |t-c|<\delta$ we have

$$\left|\frac{F(t)-F(c)}{t-c}-f(c)\right| = \left|\frac{\int_c^t f(x)\,dx - \int_c^t f(c)\,dx}{t-c}\right| \le \left|\frac{\int_c^t |f(x)-f(c)|\,dx}{c-t}\right| < \varepsilon$$

Thus $\displaystyle F'(c)=\lim_{t\to c}\frac{F(t)-F(c)}{t-c}$ exists and $F'(c)=f(c)$ $\square$

Example $f:[-1,1]\to\mathbb{R}$ $f(x)=\begin{cases}0 & x\in[-1,0]\\ 1 & x\in(0,1]\end{cases}$

$$F(t)=\int_{-1}^{t} f(x)\,dx=\begin{cases}0 & t\in[-1,0]\\ t & t\in(0,1]\end{cases}$$

$F(t)$ is continuous on $[-1,1]$, differentiable on $[-1,0)\cup(0,1]$ but not differentiable at $t=0$.

Corollary Every continuous function $f:[a,b]\to\mathbb{R}$ has an antiderivative

Proof By Theorem 4.7, $F(t)=\int_a^t f(t)\,dt$ is an antiderivative of f □

Definition 4.8 If F is an antiderivative of f, we define

$\int f(x)\,dx = F(x)+c$, the indefinite integral of f

(in contrast to the definite Riemann integral $\int_a^b f(x)\,dx$)

Theorem 4.9 If f and g have antiderivatives, then so do $f+g$ and cf for $c\in\mathbb{R}$, and

$$\int (f(x)+g(x))\,dx=\int f(x)\,dx+\int g(x)\,dx \qquad \int c f(x)\,dx = c\int f(x)\,dx$$

Proof (a) $F'=f$ $G'=g$ imply $(F+G)'=F'+G'=f+g$

Thus $\int f(x)+g(x)\,dx = F(x)+G(x) = \int f(x)\,dx+\int g(x)\,dx$ □

(b) cf analogous