Theorem 28 Let $f : [a,b] \to \mathbb{R}$ be bounded. f is Riemann integrable if and only if

$$\forall \varepsilon > 0 \; \exists P \in \mathcal{P} : \quad U(f,P) - L(f,P) < \varepsilon$$

Proof "$\Rightarrow$" Let f be R-integrable and

$$A = \sup_{P \in \mathcal{P}} L(f,P) = \inf_{P \in \mathcal{P}} U(f,P)$$

Then for a given $\varepsilon > 0$ there exist $P_1, P_2 \in \mathcal{P}$ s.t.

$$A - \frac{\varepsilon}{2} < L(f,P_1) \quad \text{and} \quad U(f,P_2) < A + \frac{\varepsilon}{2}$$

For $P = P_1 \cup P_2$ we have

$$U(f,P) - L(f,P) \leq U(f,P_2) - L(f,P_1)$$

$$< A + \frac{\varepsilon}{2} - \left(A - \frac{\varepsilon}{2}\right) = \varepsilon$$

"$\Leftarrow$" If for any $\varepsilon > 0$ there is a $P \in \mathcal{P}$ s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

then
$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(f,P) - L(f,P) < \varepsilon$$

As ε is arbitrary,
$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

so f is R-integrable

$\square$

Examples 1) Let $f:[a,b]\to\mathbb{R}$, $f(x)=c$ be constant

for $P=\{x_0,\dots,x_n\}$: $m_i = M_i = c$ and

$$U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i = c\sum_{i=1}^{n}\Delta x_i = c(b-a)$$
$$L(f,P)=\sum_{i=1}^{n} m_i\Delta x_i = c\sum_{i=1}^{n}\Delta x_i = c(b-a)$$

so f is R-integrable with $\int_a^b f(x)\,dx = c(b-a)$

2) Let $f:[a,b]\to\mathbb{R}$, $f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$

for $P=\{x_0,\dots,x_n\}$: $m_i=0$, $M_i=1$ and

$$U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i = (b-a)$$
$$L(f,P)=\sum_{i=1}^{n} m_i\Delta x_i = 0$$

so f is not R-integrable

3) Let $f:[0,2]\to\mathbb{R}$, $f(x)=\begin{cases}0 & 0\le x<1\\ 1 & 1\le x\le 2\end{cases}$

Choose $0<x_1<1<x_2<2$ with $x_2-x_1<\varepsilon$. and $P=\{0,x_1,x_2,2\}$

Then $M_1=m_1=0$, $M_2=1$, $m_2=0$, $M_3=m_3=1$

$$U(f,P)=0(x_1-0)+1(x_2-x_1)+1(2-x_2)$$
$$L(f,P)=0(x_1-0)+0(x_2-x_1)+1(2-x_2)$$

$\int_0^2 f(x)\,dx = 1$

so $U(f,P)-L(f,P)=x_2-x_1<\varepsilon$ and f is R-integrable, with

Theorem 29   Every monotone function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.

Proof   Assume wlog that f is increasing. Then $f(a) \le f(x) \le f(b)$, so f is bounded.

Let $\varepsilon > 0$. Choose a partition P with a mesh $\sigma(P) \le \frac{\varepsilon}{f(b)-f(a)+1}$.

As f is increasing, $M_i = f(x_i)$, $m_i = f(x_{i-1})$, so that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\,\Delta x_i$$

$$\le \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\,\sigma(P)$$

$$\le \big(f(b) - f(a)\big)\,\frac{\varepsilon}{f(b)-f(a)+1} < \varepsilon$$

By Theorem 28, f is Riemann integrable. □

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Definition 30 $f : D \to \mathbb{R}$ is uniformly continuous if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall c \in D\ \forall x \in D : \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

Remark   This means that δ is chosen independently of c. Continuity was

$$\forall c \in D\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D : \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

Example $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is continuous, but not uniformly continuous.

Suppose for $\varepsilon = 1$ there exists a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |x^2 - c^2| < 1$

for all $x, c \in \mathbb{R}$. But for $c = \frac{1}{\delta}$ and $x = c + \frac{\delta}{2}$ we have $|x - c| = \frac{\delta}{2} < \delta$

and $|x^2 - c^2| = \left|\left(c + \frac{\delta}{2}\right)^2 - c^2\right| = c\delta + \frac{\delta^2}{4} = 1 + \frac{\delta^2}{4} > 1$, a contradiction

<u>Theorem 31</u> Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

<u>Proof</u> Suppose f is continuous on $[a,b]$ but <u>not</u> uniformly continuous.

Then $\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists c \in D \; \exists x \in D : |x - c| < \delta \Rightarrow |f(x) - f(c)| \geq \varepsilon$

So there exists an $\varepsilon > 0$ such that for $\delta = \frac{1}{n}$ $\exists \, c_n, x_n \in D$ with

$|x_n - c_n| < \delta = \frac{1}{n}$ but $|f(x_n) - f(c_n)| \geq \varepsilon$.

Now (this is the key step!) (c_n) contains a convergent subsequence by Bolzano-Weierstraß. Therefore there exist n_r, $r \in \mathbb{N}$ such that

(1) $\lim_{r \to \infty} c_{n_r} = d$ for some $d \in [a,b]$

(2) $\lim_{r \to \infty} x_{n_r} = d$ $\quad (\; |x_{n_r} - d| \leq |x_{n_r} - c_{n_r}| + |c_{n_r} - d| \;)$

(3) $\lim_{r \to \infty} f(c_{n_r}) = f(d)$ and $\lim_{r \to \infty} f(x_{n_r}) = f(d)$

but $\forall n : |f(x_n) - f(c_n)| \geq \varepsilon$, which is a contradiction. $\square$

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expand: (i) $\forall \varepsilon' > 0 \; \exists r' \; \forall r > r' : |c_{n_r} - d| < \varepsilon'$

(ii) also $|x_{n_r} - c_{n_r}| < \frac{1}{n_r} \leq \frac{1}{n}$

$\Big\}$ pick $\varepsilon' = \frac{1}{n}$ to get

$$|x_{n_r} - d| \leq |x_{n_r} - c_{n_r}| + |c_{n_r} - d| < \frac{1}{n} + \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Theorem 32 Every continuous function $f : [a,b] \to \mathbb{R}$ is Riemann-integrable.

Proof f is uniformly continuous, so that $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$|f(x) - f(x')| < \frac{\varepsilon}{b-a} \quad \text{for all } x, x' \in [a,b] \text{ with } |x - x'| < \delta.$$

Now choose a partition P with $\sigma(P) < \delta$.

Then on each interval I_i f assumes minimum and maximum at some $x, x' \in I_i$, so that $m_i = f(x)$, $M_i = f(x')$.

Now $|x - x'| \leq \sigma(P) < \delta$, so that $M_i - m_i = |f(x') - f(x)| < \frac{\varepsilon}{b-a}$

Thus $$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \varepsilon$$

By Theorem 28, f is Riemann-integrable. $\square$

Theorem 33 Let $f : [a,b] \to \mathbb{R}$ be bounded. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all partitions P of $[a,b]$ with $\sigma(P) < \delta$

$$U(f,P) < \overline{\int_a^b} f(x)\,dx + \varepsilon$$

and

$$L(f,P) > \underline{\int_a^b} f(x)\,dx - \varepsilon$$

Remark This means that a partition with sufficiently small mesh $\sigma(P)$ approximates both upper and lower sums well.

cut due to cancelled lectures

Proof. f bounded $\Rightarrow$ $|f(x)| \le M$ for all $x \in [a,b]$

(i) adding a point to a partition P decreases the upper sum by at most $2M\sigma(P)$

if $x_{i-1} < y < x_i$, then

$$U(f,P) - U(f, P \cup \{y\}) = M_i(x_i - x_{i-1}) - M'(x_i - y) - M''(y - x_{i-1})$$
$$\le 2M\,\Delta x_i \le 2M\sigma(P)$$

(ii) if a partition Q has r points, then

$$U(f,P) - U(f, P \cup Q) \le 2Mr\,\sigma(P)$$

Now fix $\varepsilon > 0$. Then there exists a partition Q such that

$$U(f,Q) < \int_a^{b*} f(x)dx + \frac{\varepsilon}{2}$$

For any $P \in \mathcal{P}$ we have therefore

$$U(f,P) \le U(f,P) + \underbrace{U(f,Q) - U(f,P \cup Q)}_{\ge 0}$$

$$\le 2Mr\,\sigma(P) + \int_a^{b*} f(x)dx + \frac{\varepsilon}{2}$$

Choosing $\delta = \frac{\varepsilon}{4Mr}$, we have that for all $P \in \mathcal{P}$ with $\sigma(P) < \delta$

$$U(f,P) < 2Mr \frac{\varepsilon}{4Mr} + \int_a^{b*} f(x)dx + \frac{\varepsilon}{2} = \int_a^{b*} f(x)dx + \varepsilon$$

A similar proof works for $L(f,P)$.

Theorem 34 Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable.

If $P_n \subset \mathcal{P}$ satisfies $\lim_{n\to\infty} \sigma(P_n) = 0$ then

$$\lim_{n\to\infty} U(f, P_n) = \int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n)$$

Proof By Theorem 33, $\forall \varepsilon > 0\ \exists \delta > 0\ \forall P \in \mathcal{P}$:

$$\sigma(P) < \delta \Rightarrow \left| U(f,P) - \int_a^b f(x)\,dx \right| < \varepsilon$$

As $\lim_{n\to\infty} \sigma(P_n) = 0$, $\forall \delta > 0\ \exists N\ \forall n > N : \sigma(P_n) < \delta$

Together, $\forall \varepsilon > 0\ \exists N\ \forall n > N : \left| U(f, P_n) - \int_a^b f(x)\,dx \right| < \varepsilon$

Thus $\lim_{n\to\infty} U(f, P_n) = \int_a^b f(x)\,dx$.

Similarly $\lim_{n\to\infty} L(f, P_n) = \int_a^b f(x)\,dx$.

Remark: If you know that f is R-integrable, it suffices to choose any P_n with $\sigma(P_n) \to 0$ to compute $\int_a^b f(x)\,dx$

end of cut material

Examples

1) $f:[a,b]\to\mathbb{R}$, $f(x)=x$

f is monotone, therefore R-integrable

Choose $P_n=\{a, a+\Delta, a+2\Delta, \dots, a+n\Delta=b\}$ where $\Delta=\frac{b-a}{n}$

$\sigma(P_n)=\Delta=\frac{b-a}{n}\to 0$ as $n\to\infty$

Therefore $U(f,P_n)=\sum_{i=1}^{n}(a+i\Delta)\Delta = an\Delta+\frac{n(n+1)}{2}\Delta^2$

$=a(b-a)+\frac{1}{2}(b-a)^2\left(1+\frac{1}{n}\right)$ $\quad\Big\|\quad$ $L(f,P_n)=an\Delta+\frac{n(n-1)}{2}\Delta^2$

and $\int_a^b f(x)dx=\lim_{n\to\infty}U(f,P_n)=a(b-a)+\frac{1}{2}(b-a)^2=\frac{b^2}{2}-\frac{a^2}{2}$

2) $f:[1,a]\to\mathbb{R}$, $f(x)=\frac{1}{x}$

f is monotone, therefore R-integrable

Chose $P_n=\{1=q^0, q^1, q^2, \dots, q^n=a\}$ where $q=\sqrt[n]{a}$

$\Delta x_i = q^i-q^{i-1}=(q-1)q^{i-1}$, $\sigma(P_n)=(q-1)q^{n-1}=a\left(1-\frac{1}{\sqrt[n]{a}}\right)\to 0$ as $n\to\infty$.

Therefore $L(f,P_n)=\sum_{i=1}^{n}\frac{1}{q^i}(q-1)q^{i-1}=n\left(1-\frac{1}{q}\right)$ $\quad\Big|\quad$ $U(f,P_n)=n(q-1)$

and $\int_a^b f(x)dx=\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}n\left(1-e^{-\frac{1}{n}\log a}\right)$

$=\lim_{t\to 0}\frac{1-e^{-t\log a}}{t}=\lim_{t\to 0}\frac{e^{-t\log a}\log a}{1}=\log a$

7. Properties of the Riemann Integral

Theorem 35 Let $f : [a,b] \to \mathbb{R}$ be R-integrable.

If $[c,d] \subset [a,b]$ then f is R-integrable over $[c,d]$.

Proof Let $\varepsilon > 0$. Then there is a $P \subset \mathbb{P}$ with $U(f,P) - L(f,P) < \varepsilon$.

If we let $P' = P \cup \{c,d\} = \{x_0, x_1, \ldots, x_k = c, x_{k+1}, \ldots, x_{k+r} = d, \ldots, x_n\}$

then $$U(f,P') - L(f,P') \leq U(f,P) - L(f,P) < \varepsilon$$

Now take $P'' = \{x_k, x_{k+1}, \ldots, x_{k+r}\}$, which is a partition of $[c,d]$ with

$$U(f,P'') - L(f,P'') = \sum_{i=k+1}^{k+r} (M_i - m_i)\Delta x_i$$

$$\leq \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$= U(f,P') - L(f,P') < \varepsilon$$

Thus f is R-integrable over $[c,d]$. ∎

2 March 05

Theorem 36 Let $f : [a,b] \to \mathbb{R}$ be R-integrable over $[a,c]$ and $[c,b]$

where $a < c < b$. Then f is R-integrable over $[a,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$