6. Definition of the Riemann Integral

Let $I = [a,b]$, $a<b$ Interval.

Given $x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$,

we call $P = \{x_0, x_1, \dots, x_n\}$ a <u>partition of I</u>

We denote $I_i = [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, n$.

A partition is called <u>equidistant</u> if all I_i have equal length Δx_i

We denote the set of all partitions of I by $\mathcal{P}$.

P_2 is called a <u>refinement</u> of P_1 if $P_1 \subseteq P_2$

Two arbitrary partitions P_1 and P_2 have a common refinement, for example $P = P_1 \cup P_2$ is such a refinement. The notion of refinement defines a partial order on $\mathcal{P}$.

$\sigma(P) = \max\{\Delta x_i \mid i = 1, 2, \dots, n\}$ is called the <u>mesh</u> of P

$P_1 \subseteq P_2$ implies $\sigma(P_1) \geq \sigma(P_2)$, i.e. a refinement has a smaller mesh.

<u>Examples</u> 1) $P = \{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \dots, a + n\frac{b-a}{n} = b\}$

is an equidistant partition with $\sigma(P) = \frac{b-a}{n}$

2) $P_2 = \{0, \frac{1}{2n}, \frac{2}{2n}, \dots, \frac{2n}{2n}\}$ is a refinement of $P_1 = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$

$\sigma(P_2) = \frac{1}{2n} < \sigma(P_1) = \frac{1}{n}$

Note that $P_3 = \{0, \frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n+1}{n+1}\}$ is <u>not</u> a refinement of P_1.

Definition 25 Let $f : [a,b] \to \mathbb{R}$ be bounded and

$P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

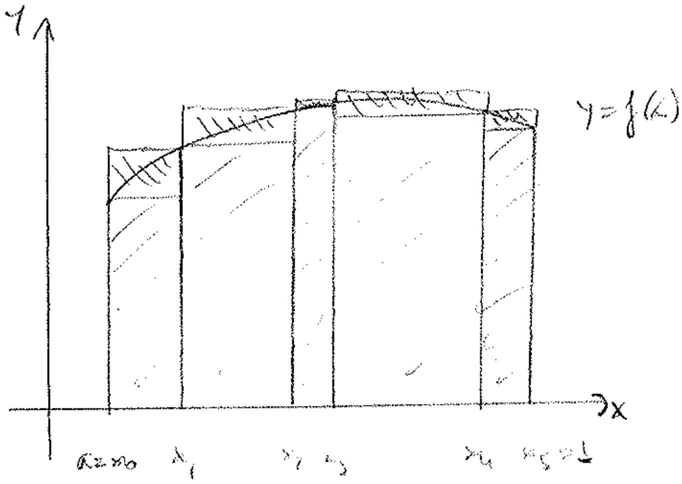
We define the upper sum of f with respect to P

$$U(f,P) = \sum_{i=1}^{n} M_i \, \Delta x_i$$

and the lower sum of f ~~with respect to P~~

$$L(f,P) = \sum_{i=1}^{n} m_i \, \Delta x_i$$

where $M_i = \sup \{ f(x) \mid x \in I_i \}$, $m_i = \inf \{ f(x) \mid x \in I_i \}$.



Geometrically, the area A between the x-axis and the graph of $f(x)$ from a to b should satisfy

$$L(f,P) \le A \le U(f,P)$$

(13 Feb 9)

Theorem 26 Let $f : [a,b] \to \mathbb{R}$ be bounded. If P_2 is a refinement of the partition P_1 then.

(1) $U(f, P_2) \le U(f, P_1)$

(2) $L(f, P_2) \ge L(f, P_1)$

Proof 1) Let $P_1 = \{x_0, x_1, \dots, x_n\}$ and $P_2 = P_1 \cup \{y\}$

If $x_{i-1} < y < x_i$ then

$$M' = \sup \{ f(x) \mid x_{i-1} \le x \le y \} \le M_i$$

and $$M'' = \sup \{ f(x) \mid y \le x \le x_i \} \le M_i$$

and $$M_i \Delta x_i = M_i (y - x_{i-1}) + M_i (x_i - y)$$

$$\ge M' (y - x_{i-1}) + M'' (x_i - y)$$

so that

$$U(f, P_1) = \sum_{\substack{j=1 \\ j \ne i}}^{n} M_j \Delta x_j + M_i \Delta x_i$$

$$\ge \sum_{\substack{j=1 \\ j \ne i}}' M_j \Delta x_j + M'(y - x_{i-1}) + M''(\cancel{x_i} - y)$$

$$= U(f, P_2)$$

2) Let P_2 be an arbitrary refinement of P_1. Then P_2 is obtained from P_1 by adding a finite number of points y_j, creating a chain of partitions $P_1 = Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_r = P_2$

and by 1) $$U(f, P_1) \ge U(f, Q_1) \ge \dots \ge U(f, P_2)$$

A similar argument leads to $L(f, P_2) \ge L(f, P_1)$

$\square$

<u>Corollary</u> Let P_1, P_2 be partitions of $[a,b]$. Then

$$L(f, P_1) \leq U(f, P_2)$$

<u>Proof</u> Let $P = P_1 \cup P_2$ be a common refinement of P_1 and P_2.

Then $L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$ $\square$

<u>Corollary</u> $\{U(f,P) \mid P \in \mathcal{P}\}$ is bounded below

$\{L(f,P) \mid P \in \mathcal{P}\}$ is bounded above

We define the upper integral of f

$$\int_a^{b*} f(x)\,dx = \inf \{U(f,P) \mid P \in \mathcal{P}\}$$

and the lower integral of f

$$\int_{a*}^{b} f(x)\,dx = \sup \{L(f,P) \mid P \in \mathcal{P}\}$$

These quantities exist for bounded $f : [a,b] \to \mathbb{R}$, and we have

$$\int_{a*}^{b} f(x)\,dx \leq \int_a^{b*} f(x)\,dx$$

<u>Definition 27</u> A bounded function $f : [a,b] \to \mathbb{R}$ is <u>Riemann integrable</u>

if the upper and lower integral of f agree. The quantity

$$\int_a^b f(x)\,dx = \int_{a*}^{b} f(x)\,dx = \int_a^{b*} f(x)\,dx$$

is called the <u>Riemann integral of f over $[a,b]$</u>